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A NEW PROOF OF D. POPESCU'S THEOREM ON SMOOTHING OF RING HOMOMORPHISMS

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Dedicated to Professor H. Hironaka on the occasion of his sixtieth birthday

Contents

§1. Introduction

In this paper we give a new proof, and some strengthenings of the following theorem of D. Popescu:

Theorem 1.1 ([1], [13]–[16] and [18])**.** *Let $A \xrightarrow{\sigma} B$ be a homomorphism of noetherian rings. The homomorphism σ is regular if and only if B is a filtered inductive limit of smooth A-algebras of finite type.*

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(See [10, Chapter 11 and Chapter 13, (33.A), p. 249] for the definition of regular and smooth homomorphisms and §2 for a discussion of some of their main properties.)

"If" is well known: see Popescu's argument in [18, Lemma 1.4], or apply André's theorem (Property 2.8 and Corollary 2.9 below) and the fact that André homology commutes with direct limits ([18, Lemma 3.2] and [2, Chapter III, Proposition 35]). Our main interest is in proving "only if".

Theorem 1.1 can be restated as follows. To say that B is a filtered inductive limit of smooth finite type A-algebras is equivalent to saying that any commutative diagram

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d, "\tau"'] & B \\ C \arrow[ur, "\rho"'] & \end{tikzcd} \tag{1.1}$$

where C is a finitely generated A-algebra, can be extended to a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d, "\tau"'] & B \\ C \arrow[ur, "\rho"'] \arrow[r, "\phi"'] & D \arrow[u, "\psi"'] \end{tikzcd} \tag{1.2}$$

where D is a smooth finitely generated A-algebra. Theorem 1.1 asserts that such an extension exists whenever σ is a regular homomorphism.

There are two refinements of Theorem 1.1, conjectured by M. Artin in [4] at the same time with Theorem 1.1. The first [4, p. 225, Conjecture 2] is to require that the map ϕ in the diagram (1.2) be smooth wherever possible (roughly speaking, away from the non-smooth locus of C). In our proof, this requirement is satisfied by construction. Thus, what we actually prove is the following stronger version of Theorem 1.1.

For an A-algebra C of finite type, let $H_{C/A}$ denote the Jacobian ideal of C over A (see Definition 2.11 for the definition of $H_{C/A}$). The ideal $H_{C/A}$ defines the non-smooth locus of C over A; in other words, for a prime ideal P of C, C_P is smooth over A if and only if $H_{C/A} \not\subset P$—see §2 for details.

Theorem 1.2. *Consider a commutative diagram (1.1), where σ is a regular homomorphism of noetherian rings. Then there exists a commutative diagram (1.2) such that D is smooth of finite type over A and $H_{C/A}B \subset \sqrt{H_{D/C}B}$.*

This gives an affirmative answer to Conjecture 2 of [4].

The second direction in which Theorem 1.1 can be strengthened is

Problem 1.3 ([4, p. 224]). *Assume that the homomorphism ρ in diagram (1.1) is injective. Does there exist a diagram (1.2) with D smooth of finite type over A, such that ψ is also injective? In other words, is B a filtered inductive limit of smooth finitely generated A-subalgebras?*

Note that Problem 1.3 makes sense even if σ itself is not injective. See the Appendix for a discussion of non-injective regular homomorphisms.

Problem 1.3 is stated in [4] in the case when A is a field and $B = A[[x_1, \dots, x_n]]$ is a formal power series ring over A. In fact, it turns out to have an affirmative answer for a wide class of regular homomorphisms. Namely, in §10 we give an affirmative solution to Problem 1.3 assuming that A is reduced and for any minimal prime Q of B, $\frac{B}{Q}$ has infinite transcendence degree over $\kappa(Q \cap A)$ (in particular, whenever A is a reduced noetherian ring and $B = A[[x_1, \dots, x_n]]$, or when A is reduced, essentially of finite type over a field or $\mathbb{Z}$ and B is the completion of A along a non-zero ideal—see §10 for details). We give an example to show that the assumption of infinite transcendence degree is necessary. We also give an example of a diagram (1.1) with ρ injective and A non-reduced, such that there does not exist a diagram (1.2) with ψ injective and $H_{C/A}B \subset \sqrt{H_{D/C}B}$. This shows that the hypothesis that A is reduced is necessary, at least for this method of proof.

Two special cases of Theorem 1.1 are of particular interest for applications. The first is the case when (A, I) is a Henselian pair such that the I-adic completion homomorphism $\sigma : A \to \hat{A}$ is regular (this is the case whenever A is excellent and, more generally, whenever A is a G-ring [10, (33.A) and (34.A)]). Applying Theorem 1.1 to σ yields the general form of Artin approximation theorem (which was already known to follow from D. Popescu's theorem and is included here mainly for completeness), as well as a generalized version of the nested approximation theorem and B. Teissier's nested smoothing theorem (see §11).

The second special case of interest is the case when B is a regular local ring and A is a field or a Dedekind domain, contained in B, such that the inclusion map $A \to B$ is a regular homomorphism (if B is equicharacteristic, we may take A to be the prime field of B). Theorem 1.1, applied in this case, yields a positive answer to the Bass–Quillen conjecture in the equicharacteristic case, as well as to several related conjectures on freeness of projective modules (see [18] for details).

Conventions. All the rings in this paper will be commutative with 1. We will denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{N}_0$ the set of non-negative integers.

For an ideal I, $\sqrt{I}$ will denote the radical of I. Let $\sigma : A \to B$ be a homomorphism of rings. If I is an ideal of A, we write IB for $\sigma(I)B$. If P is a subset of B, we write $P \cap A$ for $\sigma^{-1}(P)$. The module of relative Kähler differentials of B over A will be denoted by $\Omega_{B/A}$. An A-algebra B will be said to be **of finite type** if it is finitely presented over A, **essentially of finite type over** A if B is a localization of a finite type A-algebra. A **free** A-algebra is a polynomial ring over A, on an arbitrary (not necessarily finite) set of generators. For a prime ideal P in a ring A, $\kappa(P)$ will denote the residue field of A_P. If A is a ring and M an A-module, S_AM will stand for the symmetric algebra over M. If S is a multiplicative subset of A, M_S will denote the localization of M with respect to S, that is, $M_S = M \otimes_A A_S$. Similarly, if P is a prime ideal of A, we will write M_P for $M \otimes_A A_P$. If m is an ideal in a ring A, $Ann_A m^\infty$ will stand for $\bigcup\limits_{i=1}^{\infty} Ann_A m^i$. Let A be a ring and $u = (u_1, \dots, u_n)$ independent variables. Given $r \in \mathbb{N}_0$ and a subset $g = \{g_1, \dots, g_r\} \subset A[u]$, Δ_g will denote the ideal of $A[u]$ generated by all the $r \times r$ minors of the matrix $\left(\frac{\partial g}{\partial u}\right) \equiv \left(\frac{\partial g_i}{\partial u_j}\right)_{\substack{1 \le i \le r \\ 1 \le j \le n}}$. If v is a subset of the variables u, $\Delta_{g,v}$ will stand for the ideal generated by all the $r \times r$ minors of the matrix $\left(\frac{\partial g}{\partial v}\right)$. If $g_1, \dots, g_r$ are linear homogeneous, we will denote by Δ_g^0 the ideal of A generated by the $r \times r$ minors of $\left(\frac{\partial g_i}{\partial u_j}\right)_{\substack{1\le i\le r\\ 1\le j\le n}}$. In the case $r = 0$, we adopt the convention that the determinant of the empty matrix is 1.

We now outline the proof of Theorem 1.2. Our proof (as well as Popescu's original proof of Theorem 1.1) starts out with the following two observations, due to D. Popescu. Consider a diagram (1.1).

Lemma 1.4. *Suppose that $H_{C/A}B = B$. Then there exists a diagram (1.2) with D smooth of finite type over A and such that $H_{C/A}D = H_{D/C} = D$.*

Proof. Let $a_1, \dots, a_n$ be a set of generators of $H_{C/A}$. Then there exist $b_1, \dots, b_n \in B$ with $\sum\limits_{i=1}^{n} \rho(a_i) b_i = 1$. Let $T_1, \dots, T_n$ be new variables and let $D = \frac{C[T_1, \dots, T_n]}{\left(\sum_{i=1}^n a_i T_i - 1\right)}$. Define the map $\psi : D \to B$ by $\psi(T_i) = b_i$, $1 \le i \le n$. We obtain a commutative diagram (1.2). Since D is defined over C by the single equation $\sum\limits_{i=1}^{n} a_i T_i - 1 = 0$, we have $H_{D/C} = (a_1, \dots, a_n)D = D = H_{C/A}D$; in particular, D is smooth over C. Also, for any prime P of D, the ideal $P \cap C$ does not contain $(a_1, \dots, a_n) = H_{C/A}$, hence $C_{P\cap C}$ is smooth over A, and thus D_P is smooth over A by the transitivity of smoothness [10, Chapter 11, (28.E), p. 201]. Thus D is smooth over A and the desired diagram (1.2) is constructed. $\square$

Thus, to prove Theorem 1.1 and Theorem 1.2 it is sufficient to construct a diagram (1.2), with D an A-algebra of finite type, such that $H_{D/A}B = B$. Therefore we may assume that $H_{C/A}B \subsetneqq B$ in (1.1). Let P be a minimal prime of $H_{C/A}B$. The second observation is that to prove Theorem 1.1, it is sufficient to prove the following theorem.

Theorem 1.5. *There exists a diagram (1.2) such that:*
(1) $\sqrt{H_{C/A}B} \subset \sqrt{H_{D/A}B}$.
(2) $H_{D/A}B \not\subset P$.

Indeed, suppose Theorem 1.5 is known. Since $H_{C/A}B \subset P$ by definition of P, (1) and (2) of Theorem 1.5 imply that $\sqrt{H_{C/A}B} \subsetneqq \sqrt{H_{D/A}B}$. Apply Theorem 1.5 to the A-algebra D instead of C, and iterate the procedure. By noetherian induction on $\sqrt{H_{C/A}B}$, after finitely many steps we will arrive at the situation when $H_{D/A}B = B$, and Theorem 1.1 will follow from Lemma 1.4.

In this paper, the A-algebra D constructed in Theorem 1.5 will satisfy the additional condition

$$\sqrt{H_{C/A}B} \subset \sqrt{H_{D/C}B} \tag{1.3}$$

(in fact, (1.3) implies (1) of Theorem 1.5 by Property 2.16 below). Using this stronger version of Theorem 1.5 in the above noetherian induction argument yields Theorem 1.2.

Our original idea for the proof of Theorem 1.1 came from Lazard's theorem (recalled in §3), which says that an A-module is flat if and only if it is a filtered inductive limit of free finitely generated A-modules, and the realization that Theorem 1.1 is for A-algebras what Lazard's theorem is for A-modules (the analogy between regular homomorphisms $A \to B$ of rings and flat A-modules is discussed in more detail in §2). In fact, this is more than an analogy: Lazard's theorem is used in a crucial way in the proof of Theorems 1.5 and 1.2. Namely, in §3 we use

Lazard's theorem to deduce Theorem 1.2 and Theorem 1.5 in the case when C is defined over A by linear homogeneous equations by writing $C = S_A M$, with M a finite A-module (Proposition 3.4). Since B is A-flat, the existence of a diagram (1.2) is given by Lazard's theorem. Indeed, consider a presentation

$$L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \to 0 \tag{1.4}$$

of M, where the L_i are free of finite rank. Let $d_1^* : L_0^* \to L_1^*$ be the dual of d_1. Take a free A-module K which maps surjectively onto $Ker(d_1^*)$. We get an exact sequence

$$K \xrightarrow{\alpha} L_0^* \xrightarrow{d_1^*} L_1^*. \tag{1.5}$$

Since B is flat over A, the sequence $K \otimes_A B \xrightarrow{\alpha\otimes B} L_0^* \otimes_A B \xrightarrow{d_1^*\otimes B} L_1^* \otimes B$ obtained by tensoring (1.5) with B, is also exact. Let $u \in L_0^* \otimes B$ be the element corresponding to $\rho \circ d_0$ under the identification $L_0^* \otimes B \cong Hom_A(L_0, B)$. We have $u \in Ker(d_1^* \otimes B)$, hence

$$u \in Im(\alpha \otimes B). \tag{1.6}$$

To construct a diagram (1.2) for the given A-algebra C, we need flatness of B over A only to conclude (1.6). Thus (1.2) can be constructed even when B is not flat over A, provided (1.6) holds. We prove Lazard's theorem and Proposition 3.4 in this slightly stronger form: in §§8–9 we will apply it in a situation when B is not necessarily flat over A. We note that the diagram (1.2) which we construct satisfies the condition (1.3).

With a view to Theorem 1.5, we also prove the following version of the linear homogeneous case (Proposition 3.6). Write $C = \frac{A[u]}{(f)}$, $f = (f_1, \dots, f_r)$. Let m be a prime ideal of A such that $mA_m \subset \sqrt{\Delta_f^0 A_m}$. Then there exists a diagram (1.2) such that $D \otimes_A A_m$ is smooth over A and $mD \subset H_{D/C}$ (in other words, we can resolve the singularities at m by a map $C \to D$ which is smooth away from $V(m)$). Finally, we push Proposition 3.6 even further to prove the following **linearized case** of Theorem 1.5. §§4–9 are spent reducing the general case of Theorem 1.5 to the linearized one, thus proving Theorem 1.5 in its full generality.

Proposition 1.6 (the linearized case of Theorem 1.5)**.** *Consider a diagram (1.1). Assume that there exists a commutative diagram*

$$\begin{tikzcd}
A^* \arrow[r, "\sigma^*"] \arrow[d] & B & & \\
C \otimes_A A^* \arrow[r, "\pi^*"] & C_N^* \arrow[u, "\rho_N^*"] & C^* \arrow[l, "\lambda^*"'] \arrow[ul] & C_0^* \arrow[l, "g^*"'] \arrow[ull]
\end{tikzcd} \tag{1.7}$$

compatible with (1.1), where:

(1) A^* *is an A-algebra essentially of finite type, satisfying*

$$P \subsetneqq \sqrt{H_{A^*/A} B}. \tag{1.8}$$

(2) *Let* $m^* = P \cap A^*$. *Then*

$$\sqrt{m^* B} = P. \tag{1.9}$$

(3)

$$P \subset \sqrt{H_{C_N^*/(C\otimes_A A^*)}B}. \tag{1.10}$$

(4) *Let $m = P\cap A$ and $I = Ker\ \lambda^*$. There is a positive integer N such that:*

$$I_{P\cap C^*} \subset (m^*)^N C^*_{P\cap C^*}, \tag{1.11}$$

$$m^* C^*_{P\cap C^*} \subset \sqrt{(I^2 : I)C^*_{P\cap C^*}} \qquad \textit{and} \tag{1.12}$$

$$Ann_{A_m}(m^\infty A_m) \cap m^N A_m = (0). \tag{1.13}$$

(5) *C_0^* is defined over A^* by linear homogeneous equations; write $C_0^* = S_{A^*}M = \frac{A^*[u]}{(f)}$, $f = (f_1,\dots,f_r)$.*

(6) *Condition (1.6) (with A replaced by A^*) holds for M.*

(7) *$m^*A^*_{m^*} \subset \sqrt{\Delta_f^0 A^*_{m^*}}$.*

(8)

$$H_{C^*/C_0^*} \not\subset P\cap C^*. \tag{1.14}$$

Then the conclusion of Theorem 1.5 and (1.3) hold.

Proposition 1.6 is proved by applying Proposition 3.6 to the A^*-algebra C_0^*. We obtain a C_0^*-algebra D_0 mapping to B, such that

$$H_{D_0/A^*}\cap A^* \not\subset m^* \quad \text{and} \tag{1.15}$$

$$m^*D_0 \subset H_{D_0/C_0^*}. \tag{1.16}$$

Put $D := D_0 \otimes_{C_0^*} C_N^*$. The algebra D maps to B; this gives a diagram (1.2). We show that (1.8), (1.11)–(1.14) and (1.15) imply that $D_{P\cap D}$ is smooth over A; this gives (2) of Theorem 1.5. Moreover, from (1.8), (1.10), (1.16) and transitivity of smoothness, we get (1.3) and hence Theorem 1.5 (1).

Note that if B is local with maximal ideal P, then, localizing by any element $x \in H_{D/A}\setminus (P\cap D)$, we get that D_x is smooth over A, proving Theorem 1.2.

To prove Theorem 1.5 from Proposition 1.6, it remains to construct a diagram (1.7) satisfying conditions (1)–(8). This is accomplished in §§4–9, first under some additional assumptions about A and B and then in full generality. We now outline the construction of (1.7) in §§4–9.

Definition 1.7. Let C be an A-algebra of finite type, M a prime ideal of C and $m = M\cap A$. We say that C is an **almost complete intersection** over A at M if the following conditions hold:

(1) $mC_M \subset H_{C_M/A}$.

(2) There is a presentation $C = \frac{A[u]}{I}$ such that the restriction of $\frac{\mathcal{I}}{\mathcal{I}^2}$ to Spec $C_M \setminus V(mC_M)$ is a trivial vector bundle (here $\mathcal{I}$ denotes the coherent ideal sheaf on Spec C, given by I).

(3) Let $r = \text{rk}\ \frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\text{Spec } C_M\setminus V(mC_M)}$. Then there exist $f_1,\dots,f_r \in I$ whose natural images in $\Gamma\left(\text{Spec } C_M\setminus V(mC_M), \frac{\mathcal{I}}{\mathcal{I}^2}\right)$ generate $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\text{Spec } C_M\setminus V(mC_M)}$.

Note that (1)–(3) of Definition 1.7 are equivalent to saying that there exist $f_1,\dots,f_r \in I$ such that

$$mC_M \subset \sqrt{\Delta_f C_M} \qquad \text{and} \tag{1.17}$$

$$mC_M \subset \sqrt{(I^2+(f)) : I)C_M}, \tag{1.18}$$

where $f = (f_1, \dots, f_r)$ (see the definition of $H_{C/A}$ and Remark 2.15 below). Note also that there are two special situations in which C is an almost complete intersection over A: one is when $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\text{Spec } C_M \setminus V(mC_M)} = 0$ and Definition 1.7 (1) holds, the other when $V(mC_M) = \text{Spec } C_M$, that is, when m is nilpotent. In both of these cases, we will take $r = 0$ and $f = \emptyset$.

In §4, we construct the diagram (1.7) (proving Theorem 1.5 and Theorem 1.2) in the following **basic case**. Let P be a prime ideal of B and let $m = P \cap A$.

Proposition 1.8. *Assume that:*

(1) *B is a local ring with maximal ideal P.*
(2) *The map $\sigma : A \to B$ induces an isomorphism $\kappa(m) \tilde{\to} \kappa(P)$.*
(3) *$C = \frac{A[u]}{I}$ is an almost complete intersection over A at $P \cap C$.*
(4) *σ is flat and $mB = P$.*

Then there exists a diagram (1.7), satisfying conditions (1)–(8) of Proposition 1.6; in particular, Theorem 1.2 holds.

We also show (Remark 4.6) that Theorem 1.2 holds whenever assumptions (1)–(3) of Proposition 1.8 are satisfied and σ is formally smooth in the P-adic topology (this condition is weaker than σ being regular — see §2 fore more details).

Our main technique here is a transformation $C \to C_1$, with C_1 finitely generated over C, called "generalized blowing up" along an ideal $M \subset A$ (Definition 4.1). Generalized blowing up of C along M depends on the presentation $C = \frac{A[u_1,\dots,u_n]}{I}$ and the choice of generators of M, and is defined for any diagram (1.1) such that

$$\rho(u_i) \in \sigma(A) + MB, \ 1 \le i \le n. \tag{1.19}$$

For the purposes of Theorem 1.2 and conditions (1.3) and (1.10), we note that $\sqrt{MC_1} \subset H_{C_1/C}$ by definition of generalized blowing up, and that all the generalized blowings up in this paper will be along ideals M such that $H_{C/A}B \subset \sqrt{MB}$. We prove Proposition 1.8 by constructing a diagram (1.7) satisfying conditions (1)–(8) of Proposition 1.6, such that $A^* = A_m$ and $\pi^* : C \otimes_A A^* \to C^*_N = \frac{A^*[u^{(N)}]}{I_N}$ is a sequence of generalized blowings up along $m^* \equiv mA_m$. In the main part of the proof of Proposition 1.8 (Lemma 4.4) we replace (4) by a slightly more general hypothesis, in order to apply the result in §§8–9 in a situation where B will not be flat over A. §§5–9 are devoted to gradually extending the construction of the diagram (1.7) from the basic case to the general one. In §5 we consider a diagram (1.1), such that (A, m) and (B, P) are local,

$$\begin{aligned} &\frac{A}{m} \cong \frac{B}{P}, \\ &mB = P \qquad \text{and} \\ &P = \sqrt{H_{C/A}B}. \end{aligned} \tag{1.20}$$

We show, under some additional assumptions, more general than flatness of B over A (Lemma 5.4), that after a sequence $C \to C_L$ of generalized blowings up, C_L is an almost complete intersection over A at $P \cap C$. Combined with Proposition 1.8, this yields a diagram (1.7), satisfying (1)–(8) of Proposition 1.6. This proves Theorems 1.2 and 1.5 assuming that B is flat over A and conditions (1.20) hold. In §§6–7, in order to achieve (3) of Proposition 1.8, we prove the following version of the Nica–Popescu theorem (see Corollary 7.9 for the original Nica–Popescu theorem). Any formally smooth local homomorphism $\sigma : (A, m, k) \to (B, P, K)$ of local noetherian rings, with B complete, has a factorization $(A, m, k) \to (A^\bullet, m^\bullet, K) \xrightarrow{\sigma^\bullet} (B, P, K)$ such that $A^\bullet$ is a local noetherian ring, smooth over A, $\dim A^\bullet = \dim A + \dim_K H_1(k, K, K)$, the ring (B, P) is formally smooth over $(A^\bullet, m^\bullet)$ and $\sigma^\bullet$ induces an isomorphism of residue fields. By construction, $A^\bullet$ will be a filtered inductive limit of smooth finite type A-algebras. Furthermore, adjoining $\dim B - \dim A^\bullet$ independent variables to $A^\bullet$, mapping them to elements of B which induce a regular system of parameters of $\frac{B}{m^\bullet B}$ and localizing, we obtain a local noetherian ring (A', m', K), such that A' is a filtered inductive limit of smooth finite type A-algebras, (B, P) is formally smooth over (A', m') and $m'B = P$. The main interest and the main difficulty of the Nica–Popescu theorem is the case when K is not separable over k. Incidentally, this is the only step in the proof of Theorems 1.5 and 1.2 which uses the fact that the homomorphism σ is regular; only flatness of B over A is used in §§4–5.

Now let $\hat{B}$ denote the P-adic completion of B_P and let $m = P \cap A$. Applying the results of §7 to the local homomorphism $A_m \to \hat{B}$, we obtain a factorization $A_m \to A' \to \hat{B}$ as above. Applying the results of §§4–5 to $C \otimes_A A'$ and then replacing A' by a suitable A-subalgebra $A_1 \subset A'$, smooth of finite type over A, we obtain the diagram

(1.21)
$$\begin{tikzcd}
A_1 \arrow[r, "\sigma"] \arrow[d] & \hat{B} & & & \\
C \otimes_A A_1 \arrow[r, "\pi^{(L)}"] & C_L \arrow[u, "\rho_L"] \arrow[r, "\pi^{(N)}"] & C_{N+L} \arrow[ul] & \bar{C} \arrow[l, "\lambda"'] \arrow[ull] & C_0 \arrow[l, "g"'] \arrow[ulll]
\end{tikzcd}$$

satisfying (1)–(8) of Proposition 1.6. Now Proposition 1.6 yields Theorem 1.5 and Theorem 1.2 in the case when B is a *complete* local ring with maximal ideal P. In §8 we P-adically approximate (1.21) and obtain the diagram

(1.22)
$$\begin{tikzcd}
\tilde{A} \arrow[r] \arrow[d] & B_P & & \\
C \otimes_A \tilde{A} \arrow[r, "\tilde{\pi}"] & \tilde{C}_{N+L} \arrow[u] & \tilde{C} \arrow[l] \arrow[ul] & \tilde{C}_0 \arrow[l] \arrow[ull]
\end{tikzcd}$$

Here B_P is not necessarily flat over $\tilde{A}$. The key point is to take a P-adic approximation close enough so that

(1) the hypotheses of Lemmas 4.4 and 5.4 hold, so that these lemmas still apply.
(2) (1.6) still holds for $\tilde{C}_0$, so that Proposition 1.6 can be applied.

Applying Proposition 1.6 to the diagram (1.22) proves Theorem 1.5 in the case when B is local with maximal ideal P. Finally, in §9 we lift (1.22) from B_P to B and obtain a diagram (1.7) satisfying (1)–(8) of Proposition 1.6. This proves Theorem 1.5, (1.3) and thus Theorem 1.2 in its full generality. The lifting of the diagram (1.22) to (1.7) will be referred to as "delocalization". In §2 we recall some basic definitions and facts about smoothness, regularity, the Jacobian ideal, André homology and the relationships between them. We use Swan's definition of the *first* André homology and cohomology modules. André homology appears here for two reasons. First, it is one of the main ingredients in the proof of the Nica–Popescu theorem: it provides a language ideally suited for measuring the inseparability of residue field extensions, induced by regular homomorphisms. The second reason is motivational: the characterization of smoothness and regularity by the vanishing of André cohomology and homology, respectively, helps clarify the analogy between A-algebras and A-modules (particularly, between regular homomorphisms $A \to B$ and flat A-modules M). From this point of view, the André homology and cohomology modules, $H_i(A, B, W)$ and $H^i(A, B, W)$ can be viewed as the algebra analogues of $Tor_i^A(M, W)$ and $Ext_A^i(M, W)$, respectively.

Our motivation for considering generalized blowings up comes from the theory of resolution of singularities. Namely, consider a diagram (1.1) where A is a field, C a finitely generated A-algebra without zero divisors and $B = R_\nu$ a valuation ring of the field of fractions of C. Then the problem of constructing a diagram (1.2) (with ψ injective) is nothing but the problem of local uniformization of Spec C with respect to the valuation ν. Of course, the valuation ring R_ν is not, in general, noetherian. The intersection between the problems of Local Uniformization and Néron desingularization is precisely the case when B is a discrete valuation ring. In that case, Néron solved the problem by successively blowing up along the Jacobian ideal. In that sense, our proof can be regarded as a generalization of Néron's.

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§2. Formal smoothness, smoothness, Jacobian criterion and André homology

In this section we recall some basic properties of formal smoothness, smoothness, the Jacobian ideal and the first André homology and cohomology, used in the rest of the paper.

André homology. Let $\sigma : A \to B$ be a homomorphism of rings and W a B-module. Associated to this data, André [2] defines the homology and cohomology modules $H_i(A, B, W)$ and $H^i(A, B, W)$, $i \in \mathbb{N}_0$, which can be viewed as obstructions to regularity and smoothness of σ, respectively (see Properties 2.4 and 2.8 below). The definitions of André homology and cohomology, given in [2], are rather technical. For the reader's convenience, we recall Swan's ad hoc definitions of $H_i(A, B, W)$ and $H^i(A, B, W)$, $i \in \{0, 1\}$ (see Definition 2.1 below and [18, §3]), which are sufficient for our purposes. It is not too hard to prove from scratch all the facts about André homology needed in this paper, using these definitions. However, we will refrain from doing so to save space, instead giving references to André [2] and Swan [18]. The readers who are already familiar with André homology will recognize that Swan's ad hoc definitions are equivalent to those of André. We now define $H_i(A, B, W)$ and $H^i(A, B, W)$ for $i \in \{0, 1\}$, following Swan [18, §3]. Take an exact sequence

$$0 \to I \to F \xrightarrow{\pi} B \to 0, \tag{2.1}$$

where F is a localization of a free A-algebra and I is a (not necessarily finitely generated) ideal of F. (2.1) gives rise to the exact sequence

$$\frac{I}{I^2} \xrightarrow{d} \Omega_{F/A} \otimes_F B \to \Omega_{B/A} \to 0 \tag{2.2}$$

(the second fundamental exact sequence for Kähler differentials [10, Chapter 10, (26.I), Theorem 58, p. 187]). Tensoring (2.2) with W over B gives the exact sequence

$$\frac{I}{I^2} \otimes_B W \xrightarrow{d\otimes W} \Omega_{F/A} \otimes_F W \to \Omega_{B/A} \otimes_B W \to 0. \tag{2.3}$$

Taking B-homomorphisms into W in (2.2), we obtain the exact sequence

$$Hom_B\left(\frac{I}{I^2}, W\right) \underset{\partial_W}{\longleftarrow} Der_A(F,W) \leftarrow Der_A(B,W) \leftarrow 0. \tag{2.4}$$

Note that $\Omega_{F/A} \otimes_F B$ is a free B-module and $Der_A(F,B)$ is free whenever F is finitely generated over B.

Definition 2.1 (Swan [18, §3])**.** André homology modules and cohomology modules, $H_i(A,B,W)$ and $H^i(A,B,W)$ for $i \in \{0,1\}$, are defined as follows:

$$\begin{aligned} H_0(A,B,W) &= \Omega_{B/A} \otimes_B W \equiv Coker(d \otimes W),\\ H_1(A,B,W) &= Ker(d \otimes W),\\ H^0(A,B,W) &= Der_A(B,W) \equiv Ker\ \partial_W,\\ H^1(A,B,W) &= Coker\ \partial_W. \end{aligned}$$

It is immediate to show that for $i \in \{0,1\}$, $H_i(A,B,W)$ and $H^i(A,B,W)$ are independent of the presentation (2.1), and also that $H_i(A,\cdot,W)$ and $H^i(A,\cdot,W)$ are, respectively, a covariant and a contravariant functor of B (see [18, Part II, Lemma 3.1]).

Property 2.2. *Let $\sigma : A \to B$ be a* surjective *ring homomorphism, let $I = Ker\ \sigma$ and let W be a B-module. We have $H_0(A,B,W) = H^0(A,B,W) = 0$ and $H_1(A,B,W) \cong \frac{I}{I^2} \otimes_B W$.*

Proof. The exact sequence $0 \to I \to A \xrightarrow{\sigma} B \to 0$ gives a presentation of B as an A-module. Since $\Omega_{A/A} = 0$, the result follows immediately from definitions. □

Property 2.3 (the Jacobi–Zariski sequence)**.** *Let $F \to B$ be a homomorphism of A-algebras and let W be a B-module. There are two natural exact sequences:*

$$\begin{aligned} H_1(A,F,W) \to H_1(A,B,W) \to H_1(F,B,W) \to H_0(A,F,W)&\\ \to H_0(A,B,W) \to H_0(F,B,W) \to 0& \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} 0 \to H^0(F,B,W) \to H^0(A,B,W) \to H^0(A,F,W)&\\ \to H^1(F,B,W) \to H^1(A,B,W) \to H^1(A,F,W).& \end{aligned} \tag{2.6}$$

Proof. See [18, Part II, Theorem 3.3]. Again, Swan only proves the Jacobi-Zariski sequence for homology with coefficients in B. To get (2.5), tensor everything in Swan's diagram (*) with W; to get (2.6), take homomorphisms into W. □

Homological characterizations of smoothness and regularity. Let $\sigma : A \to B$ be a homomorphism of rings.

Property 2.4. *Consider the following conditions:*

(1) *B is smooth over A.*

(2) *The sequence*

$$0 \to \frac{I}{I^2} \xrightarrow{d} \Omega_{F/A} \otimes_F B \to \Omega_{B/A} \to 0 \tag{2.7}$$

is split exact (this condition is sometimes expressed by saying that the homomorphism d has a left inverse).

(3) *$H_1(A, B, B) = 0$ and $\Omega_{B/A}$ is a projective B-module.*

(4) *$H^1(A, B, W) = 0$ for all B-modules W.*

(5) *$H_1(A, B, B) = 0$ and $\Omega_{B/A}$ is a flat B-module.*

(6) *$H_1(A, B, W) = 0$ for all B-modules W.*

We have the implications: (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\implies$ (5) $\iff$ (6). Suppose, in addition, that either $\frac{I}{I^2}$ or $\Omega_{B/A}$ is a finitely generated B-module. Then we also have (5) $\implies$ (3), so that all the conditions (1)–(6) are equivalent. Finally, B is étale over A if and only if $H_1(A, B, B) = \Omega_{B/A} = 0$.

Proof. The implications (3) $\iff$ (4) $\implies$ (5) $\iff$ (6), as well as the equivalence (3) $\iff$ (5) in case $\frac{I}{I^2}$ or $\Omega_{B/A}$ is a finitely generated B-module, are immediate from definitions; for the rest, see [18, Theorem 3.4]. □

Note that (1), (3) and (4) of Property 2.4 do not depend on the choice of the presentation (2.1). Thus, in particular, Property 2.4 says that if (2) holds for one presentation (2.1), then it also holds for any other presentation.

Corollary 2.5. *Let $\sigma : A \to B$ be a homomorphism of rings. Then B is smooth over A if and only if B_P is smooth over A for every prime ideal P of B.*

Proof. Use the notation of (2.1). For a prime ideal P of B, $0 \to I_{P \cap F} \to F_{P \cap F} \to B_P$ is a presentation of B_P. The sequence (2.7) is split exact if and only if it is split exact after tensoring with B_P for all $P \in \operatorname{Spec} B$, and the result follows. □

Property 2.4 says that acyclicity in cohomology characterizes smoothness. On the other hand, conditions (5) and (6) — acyclycity in homology — characterize regular homomorphisms, as we now explain. We start with the case when A is a field.

Proposition 2.6. ([8, EGA 0_{IV} (22.5.8)], [10, (39.C), Theorem 93] and [2, Lemma III.21, Corollary VII.27 and Proposition XVI.17]) *Let $(A, m, k) \to (B, M, K)$ be a local homomorphism of local noetherian rings. The following conditions are equivalent:*

(1) *$H_1(A, B, K) = 0$.*

(2) *B is formally smooth over A in the M-adic topology.*

If A is a field, then (1) and (2) are also equivalent to saying that B is geometrically regular over k.

Proposition 2.7. (Grothendieck [8, Theorem (19.7.1)], [2, Proposition XV.19, p. 211]) *Let $(A, m, k) \to (B, P, K)$ be a local homomorphism of local noetherian rings. The following conditions are equivalent:*

(1) *B is formally smooth over A in the P-adic topology.*
(2) *B is flat over A and $\frac{B}{mB}$ is geometrically regular over k.*

Property 2.8 (André's theorem [2, Theorem 30, p. 331])**.** *Let $\sigma : A \to B$ be a homomorphism of noetherian rings. Then the following conditions are equivalent:*

(1) *σ is regular.*
(2) *(6) of Property 2.4 holds.*
(3) *$H_1(A, B, \kappa(P)) = 0$ for any prime ideal P of B.*
(4) *For every prime ideal $P \subset B$, B_P is formally smooth over A with respect to the P-adic topology.*

There are two situations in which acyclicity in homology and cohomology are the same thing:

Corollary 2.9. *Let $\sigma : A \to B$ be a homomorphism of noetherian rings. Assume that either B is a field or B is essentially of finite type over A. Then B is smooth over A if and only if σ is regular.*

Proof. If B is a field, then $\Omega_{B/A}$ is a B-vector space. If B is essentially of finite type over A, then $\Omega_{B/A}$ is a finite B-module. In either case, $\Omega_{B/A}$ is projective if and only if it is flat. Now the corollary follows from Properties 2.4 and 2.8. □

Remark 2.10. Let $A_1 \xrightarrow{\sigma_1} A_2 \xrightarrow{\sigma_2} A_3$ be ring homomorphisms and W an A_3-module such that $H_1(A_1, A_3, W) = \Omega_{A_2/A_1} \otimes_{A_2} W = 0$. Then $H_1(A_2, A_3, W) = 0$ (this follows immediately from the Jacobi–Zariski sequence).

Assume that A_1, A_2 and A_3 are noetherian, that $\sigma_2 \circ \sigma_1$ is a regular homomorpism and that $\Omega_{A_2/A_1} = 0$. Then σ_2 is a regular homomorpism (this follows from the above and the equivalence of (1)–(3) of Property 2.8).

The Jacobian criterion. Let A be a ring and C an A-algebra, essentially of finite type over A. Fix a presentation

$$C = \frac{A[u_1, \dots, u_n]_S}{I}, \tag{2.8}$$

where I is a finitely generated ideal of $A[u_1, \dots, u_n]$ and S a multiplicative subset of $A[u_1, \dots, u_n]$, disjoint from I. Choose a base $f = (f_1, \dots, f_l)$ of I.

Definition 2.11 (Elkik [7] and H. Hironaka)**.** The **Jacobian ideal** of C over A, denoted $H_{C/A}$, is the ideal $H_{C/A} := \sqrt{\sum_g \Delta_g((g) : I)C}$, where g ranges over all the subsets of $\{f_1, \dots, f_l\}$.

Remark 2.12. Apparently, the definition of $H_{C/A}$ depends on the presentation (2.8) and on the choice of a base for I. Property 2.13 below says that $H_{C/A}$ is the defining ideal of the non-smooth locus of C over A and therefore depends only on C itself, not on the particular presentation nor on the choice of f.

Property 2.13. *The ideal $H_{C/A}$ defines the non-smooth locus of C over A. In other words, for a prime ideal of $P \subset C$, C_P is smooth over A if and only if $H_{C/A} \not\subset P$. We have*

$$H_{C/A} = \{x \in C \mid C_x \text{ is smooth over } A\} \tag{2.9}$$

(remember that the zero ring is smooth over anything!). In particular, C is smooth over A if and only if $H_{C/A} = C$.

Proof. Let $F = A[u_1, \dots, u_n]_S$ in (2.8). We have $\Omega_{F/A} \cong \bigoplus_{i=1}^{n} F du_i$ and $\Omega_{F/A} \otimes_F C \cong \bigoplus_{i=1}^{n} C du_i$. First, suppose that C is local with maximal ideal P and residue field K.

Lemma 2.14. *C is smooth over A if and only if there exist $g_1, \dots, g_r \in \{f_1, \dots, f_l\}$ as above such that $I = (g_1, \dots, g_r)$ and $\Delta_g C = C$.*

Proof. By Property 2.4 (1) $\Longleftrightarrow$ (2), C is smooth over A if and only if the sequence

$$0 \to \frac{I}{I^2} \xrightarrow{d} \bigoplus_{i=1}^{n} C du_i \to \Omega_{C/A} \to 0 \tag{2.10}$$

is split exact. Since C is local, (2.10) is split exact if and only if it is a split exact sequence of *free* modules. This happens if and only if there is a subset $g = \{g_1, \dots, g_r\} \subset \{f_1, \dots, f_l\}$ which freely generate $\frac{I}{I^2}$ and such that $dg_1, \dots, dg_r$ are K-linearly independent modulo P. The latter condition says precisely that the matrix $\left(\frac{\partial g_i}{\partial u_j}\right)_{\substack{1 \le i \le r \\ 1 \le j \le n}}$ has rank r modulo P, that is, $\Delta_g C = C$. Now, if such a subset g exists, then $g_1, \dots, g_r$ generate I by Nakayama's lemma. Conversely, if there is a subset $g = \{g_1, \dots, g_r\} \subset \{f_1, \dots, f_l\}$ such that $\left(\frac{\partial g_i}{\partial u_j}\right)_{\substack{1 \le i \le r \\ 1 \le j \le n}}$ has rank r modulo P and $(g_1, \dots, g_r) = I$, then it is immediate to check that the images of $g_1, \dots, g_r$ *freely* generate $\frac{I}{I^2}$. This completes the proof. $\square$

Now drop the hypothesis that C is local. Since we have not yet proved that $H_{C/A}$ is independent of presentation, we will provisionally talk about $H_{C/A}$ with respect to the given presentation. Because we are assuming that I is finitely generated, the operation : commutes with localization. Thus $H_{C/A}$ localizes well: take any multiplicative subset $S \subset C$. Let $R := S \cap F$ and define $H_{C_S/A}$ using the presentation $C_S = \frac{F_R}{IF_R}$. We have a canonical isomorphism $(H_{C/A})_S \cong H_{C_S/A}$. For a prime ideal $P \subset C$, apply Lemma 2.14 to the local ring C_P (again, we use the presentation of C_P, obtained from (2.8) by localization). By Lemma 2.14, C_P is smooth over A if and only if there exists a subset $g = \{g_1, \dots, g_r\} \subset \{f_1, \dots, f_l\}$ such that

$$(g_1, \dots, g_r) F_{P \cap F} = I_{P \cap F} \tag{2.11}$$

and

$$\Delta_g \not\subset P \cap F. \tag{2.12}$$

Now, (2.11) is equivalent to saying that $((g) : I) F_{P \cap F} = F_{P \cap F}$, and also to

$$((g) : I) C_P \not\subset P C_P. \tag{2.13}$$

Combining (2.12) and (2.13) and using the fact that both the operation of taking Δ_g and : commute with localization, we get that C_P is smooth over A if and only if $\Delta_g((g) : I) C \not\subset P$ for some g as above. Thus C_P is smooth over A if and only if $H_{C/A} \not\subset P$. This proves the first statement of Property 2.13.

Since $H_{C/A}$ is radical by definition, it equals the intersection of all the primes $P \subset B$ such that B_P is not smooth over A. Now (2.9) follows from Corollary 2.5. The last statement of Property 2.13 follows immediately. $\square$

Note, in particular, that $H_{C/A}$ is well defined, i.e. is independent of the choice of presentation (2.8) and the generators f_i.

Remark 2.15. In the situation of Property 2.13, assume that F is noetherian. Suppose that C_P is smooth over A. Let Q be a minimal prime of I, contained in $P \cap F$. We have $IF_Q = QF_Q$. Then (2.11) implies that g is a system of parameters (even a regular system of parameters) for F_Q. This shows that when F is noetherian, we may let g in the definition of $H_{C/A}$ range only over those subsets $\{g_1, \dots, g_r\}$ of $\{f_1, \dots, f_l\}$ which form a system of parameters in F_Q, for some minimal prime Q of I. In particular, if C is a complete intersection over A and f is a minimal set of generators of I, then $H_{C/A} = \sqrt{\Delta_f C}$.

Consider homomorphisms $A \xrightarrow{\sigma} B \xrightarrow{\phi} C$ of noetherian rings, where B is essentially of finite type over A and C is essentially of finite type over B. The following property describes the relationship between $H_{B/A}$, $H_{C/A}$ and $H_{B/C}$.

Property 2.16. *We have* $\sqrt{H_{C/B}H_{B/A}C} = \sqrt{H_{C/B}H_{C/A}}$.

Remark 2.17. Property 2.16 says that a prime ideal $P \subset C$, not containing $H_{C/B}$, contains $H_{B/A}C$ if and only if it contains $H_{C/A}$. In view of Property 2.13, this can be restated as follows. Take any prime ideal $P \subset C$, such that C_P is smooth over B. Then C_P is smooth over A if and only if $B_{P \cap B}$ is smooth over A.

Proof of Property 2.16. By Corollary 2.9, each of the homomorphisms σ, ϕ and $\phi \circ \sigma$ is regular if and only if it is smooth. Now Property 2.16 follows from [10, (33.B)] and Proposition 2.7 (1) $\Longrightarrow$ (2). □

Field extensions. Next, we discuss some standard results, which can be interpreted as a restriction of the above theory to homomorphisms of *fields* instead of rings. A detailed study of extensions $k \to K$ with $\dim_K H_1(k, K, K) < \infty$ will be undertaken in §6. Let $k \to K$ be a field extension. Then $\Omega_{K/k}$ and $H_1(k, K, K)$ are K-vector spaces. If K is finitely generated over k, then $\Omega_{K/k}$ and $H_1(k, K, K)$ are finite-dimensional.

Property 2.18 ([18, Corollary 5.2])**.** *Let $k \to L \to K$ be homomorphisms of fields. The first map on the left in the Jacobi–Zariski sequence (2.5) is injective. In other words, we have an exact sequence*

$$\begin{aligned} 0 \to H_1(k, L, K) \to & H_1(k, K, K) \to H_1(L, K, K) \\ & \to \Omega_{L/k} \otimes_L K \to \Omega_{K/k} \to \Omega_{K/L} \to 0. \end{aligned}$$

This result is stated in [18] with $H_1(k, L, L) \otimes_L K$ instead of $H_1(k, L, K)$. However, the two statements amount to the same thing since $H_1(k, L, L) \otimes_L K \cong H_1(k, L, K)$. In fact, we have the same exact sequence for homology with coefficients in any K-vector space W instead of K (tensor everything with W and use that W is K-flat).

Property 2.19 ([18, Corollary 5.5])**.** *K is separable over k $\iff$ K is smooth over k $\iff$ $H_1(k, K, K) = 0$ $\iff$ $H_1(k, K, W) = 0$ for any K-vector space W.*

(The last two equivalences follow from Property 2.4.)

We end this section with the local criterion of flatness.

Proposition 2.20. (Local criterion of flatness, [10, (20.C), Theorem 49, p. 146] or [18, Theorem 7.1]) *Let $\sigma : A \to B$ be a homomorphism of noetherian rings, I an ideal of A such that $IB \subset Jac(B)$ and M a finitely generated B-module. The following conditions are equivalent:*

(1) *M is flat over A.*
(2) *$\frac{M}{IM}$ is flat over $\frac{A}{I}$ and*

$$Tor_1^A\left(M, \frac{A}{I}\right) = 0. \tag{2.14}$$

(3) *$\frac{M}{IM}$ is flat over $\frac{A}{I}$ and for all $n \in \mathbb{N}$, the canonical map $\frac{I^n}{I^{n+1}} \otimes_{\frac{A}{I}} \frac{M}{IM} \to \frac{I^n M}{I^{n+1}M}$ is an isomorphism.*

Proposition 2.21. *Let A be a ring, M an A-module and $(x_1, \dots, x_n)$ a regular sequence in A, which is also a regular sequence for M. Let $I = (x_1, \dots, x_n)$. Then $Tor_1^A\left(M, \frac{A}{I}\right) = 0$.*

Proof. Straightforward induction on n. See [18, Lemma 7.5]. □

Corollary 2.22. *Let $(A, m, k) \xrightarrow{\alpha} (A', m', k') \xrightarrow{\sigma'} (B, P, K)$ be local homomorphisms of local noetherian rings. Assume that:*
(1) *Both A' and B are flat over A.*
(2) *$A'_0 := \frac{A'}{mA'}$ and $B_0 := \frac{B}{mB}$ are regular local rings.*
(3) *There exist elements $x_1, \dots, x_a \in A'$ which induce a regular system of parameters of A'_0 and whose images in B_0 can be extended to a regular system of parameters of B_0.*

Then B is flat over A' (hence faithfully flat, hence σ' is injective).

Proof. Since B is flat over A, we have $Tor_1^A(k, B) = 0$. Since A' is flat over A, we obtain

$$Tor_1^A(A'_0, B) \equiv Tor_1^A(k \otimes_A A', B) = Tor_1^A(k, B) \otimes_A A' = 0. \tag{2.15}$$

Let $\bar{x} := \bar{x}_1, \dots, \bar{x}_a$ denote the images of $x_1, \dots, x_a$ in A'_0. By our assumptions, $\bar{x}$ is a regular sequence both for A'_0 and for B_0; moreover, $\frac{A'_0}{(\bar{x})}$ is a field. Thus B_0 is flat over A'_0 by the local criterion of flatness, applied at the ideal $(\bar{x})A'_0$. Combining this with (2.15) and applying the local criterion of flatness once again, this time at the ideal mA', proves that B is flat over A'. □

Remark 2.23. Let $A \to B$ be a continuous, flat homomorphism of topological rings, with topologies defined by ideals $m \subset A$ and $P \subset B$. Let $\hat{A}$ be the m-adic completion of A, $\hat{B}$ the P-adic completion of B and $\tilde{B}$ the mB-adic completion of B. Then $\tilde{B}$ is flat over $\hat{A}$ by Proposition 2.20 (1) $\iff$ (3), and $\hat{B}$ is flat over $\tilde{B}$, being the $P\tilde{B}$-adic completion of $\tilde{B}$. Thus $\hat{B}$ is flat over $\hat{A}$.

Now let the notation and assumptions be as in Corollary 2.22 and consider the induced homomorphisms $\hat{A} \xrightarrow{\hat{\alpha}} \hat{A}' \xrightarrow{\hat{\sigma}'} \hat{B}$ between formal completions. The above considerations show that $\hat{\alpha}$ and $\hat{\sigma}' \circ \hat{\alpha}$ are flat, and hence the new triple satisfies the assumptions of Corollary 2.22. Thus $\hat{B}$ is flat (hence faithfully flat) over $\hat{A}'$; in particular, $\hat{\sigma}'$ is injective.

Remark 2.24. Let $\sigma : A \to B$ be a homomorphism of topological rings, with the respective topologies defined by ideals $P \subset B$ and $m = P \cap A$. Let $\hat{A}$ denote the m-adic completion of A and $\hat{B}$ the P-adic completion of B. Then B is formally smooth over A if and only if $\hat{B}$ is formally smooth over $\hat{A}$, if and only if $\frac{B}{P^n}$ is formally smooth over $\frac{A}{m^n}$ for all $n \in \mathbb{N}$. In particular, B is formally smooth over

A whenever $\hat{A} \cong \hat{B}$ via the natural homomorphism induced by σ (all of this is immediate from definitions).

Corollary 2.25. *Let the assumptions be as in Corollary 2.22. Assume, in addition, that B is formally smooth over A, that $A' = A[x_1, \dots, x_a]_{P \cap A[x_1,\dots,x_a]}$ and that $(m, x)B = P$*

$$\frac{B}{P} \cong \frac{A}{m}. \tag{2.16}$$

Then B is formally smooth over A'.

Proof. By Remark 2.23 the induced map $\hat{\sigma} : \hat{A}[[x]] \to \hat{B}$ is injective. On the other hand, since $(m, x)B = P$ and in view of (2.16), $\hat{\sigma}$ is surjective, hence an isomorphism. The corollary follows from Remark 2.24 (this fact, even without the assumption (2.16), also follows easily from the Jacobi–Zariski sequence for the triple $A \to A' \to B$ and the B-module K). □

§3. Smoothing of algebras defined by linear homogeneous equations

In this section we prove Theorem 1.2 in the case when C is defined over A by linear homogeneous equations. For that we do not need σ to be regular: it is sufficient to assume that σ is flat (in fact, an even weaker hypothesis will do—see Proposition 3.4). All of this is well known and is a consequence of Lazard's theorem [10, (3.A), Theorem 1 (6), p. 18] and [5, pp. 7–8]. We reproduce these results here in order to go on and prove a stronger version of them (Proposition 1.6) which will play a central role in the rest of the paper.

Let A be a ring and $\rho : M \to B$ a homomorphism of A-modules, with M finitely generated. Consider a presentation

$$L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \to 0 \tag{3.1}$$

of M, where the L_i are free and L_0 is of finite rank. Let $d_1^* : L_0^* \to L_1^*$ be the dual of d_1. Take a free A-module K which maps surjectively onto $Ker(d_1^*)$. We get an exact sequence

$$K \xrightarrow{\alpha} L_0^* \xrightarrow{d_1^*} L_1^*. \tag{3.2}$$

Consider the complex

$$K \otimes_A B \xrightarrow{\alpha\otimes B} L_0^* \otimes_A B \xrightarrow{d_1^*\otimes B} L_1^* \otimes B \tag{3.3}$$

obtained by tensoring (3.2) with B. Let $u \in L_0^* \otimes B$ be the element corresponding to $\rho \circ d_0$ under the identification $L_0^* \otimes B \cong Hom_A(L_0, B)$.

Remark 3.1. By construction, $u \in Ker(d_1^* \otimes B)$. If B is flat over A, then the sequence (3.3) is exact, so that $u \in Im(\alpha \otimes B)$.

Proposition 3.2. *Assume that $u \in Im(\alpha \otimes B)$. Then there exists a factorization*

$$M \xrightarrow{\phi} F \xrightarrow{\psi} B \tag{3.4}$$

of ρ through a free finitely generated A-module F. If A is noetherian, we can choose a factorization (3.4) with the following additional property. For any $P \in$ Spec A such that M_P is a free A_P-module, the map $\phi_P : M_P \to F_P$ induced by ϕ has a left inverse.

Proof. Choose a free finitely generated submodule K_0 of K such that $u \in (\alpha \otimes B)(K_0 \otimes B)$. Consider the complex

$$K_0 \xrightarrow{\alpha} L_0^* \xrightarrow{d_1^*} L_1^*. \tag{3.5}$$

Put $F := K_0^*$. Dualizing (3.5) and using the fact that $L_1 \subset L_1^{**}$, we get a complex $L_1 \xrightarrow{d_1} L_0 \xrightarrow{\alpha *} F$. The homomorphism α^* induces a homomorphism $\phi : M \to F$. Take an element $v \in K_0 \otimes B$ such that $(\alpha \otimes B)(v) = u$. Let $\psi \in Hom_A(F, B)$ be the element corresponding to v under the identification $Hom_A(F, B) \cong K_0 \otimes B$. Then $\psi \circ \phi = \rho$ and (3.4) is constructed. Now assume that A is noetherian. Then K is finitely generated, so we may take $K_0 = K$ in (3.5). Then F is finitely generated. Take a prime $P \subset A$ such that M_P is a free A_P-module. Then $(d_0)_P : (L_0)_P \to M_P$ has a right inverse, i.e. M_P is a direct summand of $(L_0)_P$. Then M_P^* is a direct summand of $(L_0^*)_P$, so α_P induces a surjection of free modules $K_P \to M_P^*$. Hence M_P is a direct summand of $F_P = K_P^*$, so ϕ has a left inverse. $\square$

Corollary 3.3 (Lazard's theorem)**.** *Let A be a ring and $\rho : M \to B$ a homomorphism of A-modules, where B is flat and M finitely generated. There exists a factorization (3.4) of ρ through a free finitely generated A-module F. In other words, B is a filtered direct limit of free finitely generated A-modules. If A is noetherian, we can choose (3.4) so that whenever $P \in$* Spec *A and M_P is a free A_P-module, ϕ_P has a left inverse.*

Proof. Immediate from Proposition 3.2 and Remark 3.1. $\square$

We pass to symmetric algebras in (3.4) to establish Theorem 1.2 in the case when C is defined over A by linear homogeneous equations. Let $\sigma : A \to B$ be a ring homomorphism. Consider a commutative diagram (1.1). Suppose C has the form $C = \frac{A[u_1,\ldots,u_n]}{I}$, where $I = (f_1, \ldots, f_m)$ and each f_j is a linear homogeneous equation in the u_i:

$$f_j = \sum_{i=1}^{n} a_{ij} u_i, \qquad a_{ij} \in A. \tag{3.6}$$

Then $C = S_A M$, where M is the A-module with generators $u_1, \ldots, u_n$ and relations $f_1, \ldots, f_m$. Consider a presentation (3.1) of M. Then ρ induces an A-module homomorphism $\rho|_M : M \to B$. Let K and $u \in Ker(d_1^* \otimes B)$ be as above.

Proposition 3.4. *Assume that $u \in Im(\alpha \otimes B)$ in (3.3) (this holds, in particular, whenever B is flat over A). Then there exists a commutative diagram (1.2) where D is a polynomial ring in finitely many variables over A. If, in addition, A is noetherian, there exists a diagram (1.2) such that*

$$H_{C/A} D \subset H_{D/C} \tag{3.7}$$

(remember that the Jacobian ideal $H_{D/C}$ is radical by definition!).

Remark 3.5. Since D is a polynomial ring over A, we have $H_{D/A} = D$. Then by Property 2.16, (3.7) is equivalent to saying that $\sqrt{H_{C/A} D} = H_{D/C}$.

Proof of Proposition 3.4. Let

$$M \xrightarrow{\phi_1} F \xrightarrow{\psi_1} B \tag{3.8}$$

be the factorization of $\rho|_M$, described in Proposition 3.2. Put $D := S_A F$. (3.8) induces maps $C \xrightarrow{\phi} D \xrightarrow{\psi} B$ of symmetric algebras with $\rho = \psi \circ \phi$ and hence a commutative diagram (1.2). Let P be a prime ideal of D such that $H_{C/A} \not\subset P \cap C$. Let $Q = P \cap A$. Since $H_{C/A} \not\subset P \cap C$, there exists a subset of $\{f_1, \dots, f_m\}$, say $\{f_1, \dots, f_r\}$, such that $\Delta^0_{(f_1,\dots,f_r)} \not\subset Q$ and $((f_1, \dots, f_r) : I)C \not\subset P \cap C$. Then $\frac{A_Q[u_1,\dots,u_n]}{(f_1,\dots,f_r)}$ is isomorphic to a polynomial ring over A_Q, hence all of its associated primes are extended from A_Q. Since $((f_1, \dots, f_r) : I)C \not\subset P \cap C$, $((f_1, \dots, f_r) : I)\frac{A_Q[u_1,\dots,u_n]}{(f_1,\dots,f_r)}$ cannot be contained in any proper ideal extended from A_Q. Thus $((f_1, \dots, f_r) : I)A_Q[u_1, \dots, u_n] = A_Q[u_1, \dots, u_n]$, so that $C \otimes_A A_Q = \frac{A_Q[u_1,\dots,u_n]}{(f_1,\dots,f_r)}$, which implies that M_Q is a free A_Q-module. Now assume that A is noetherian. By Proposition 3.2 we may choose (3.8) so that $(\phi_1)_Q$ has a left inverse, that is, M_Q is a direct summand of F_Q. Then $D \otimes_A A_Q$ is smooth over $C \otimes_A A_Q$, hence smooth over C. Therefore $D_P \equiv (D \otimes_A A_Q)_{(PD \otimes_A A_Q)}$ is smooth over C, so $H_{D/A} \not\subset P$, as desired. □

Next, we prove a variation of Proposition 3.4 which will be used in the proof of Theorem 1.2. Consider a diagram (1.1) where $C = \frac{A[u_1,\dots,u_n]}{(f_1,\dots,f_r)}$ is defined by linear homogeneous equations (3.6). Assume that A is noetherian. Let m be a prime ideal of A such that $mA_m \subset \sqrt{\Delta^0_f A_m}$. The point of the following proposition is that we can resolve the singularitites lying over m by a homomorphism $\phi : C \to D$, smooth away from $V(m)$, even though C itself might not be smooth over A away from $V(m)$.

Proposition 3.6. *Let M, K and $u \in Ker(d_1^* \otimes B)$ be as in Proposition 3.4. Assume that $u \in Im(\alpha \otimes B)$ in (3.3). There exists a commutative diagram (1.2) such that D is defined by linear homogeneous equations over A, $mD \subset H_{D/C}$ and $H_{D/A} \cap A \not\subset m$ (i.e. $D \otimes_A A_m$ is smooth over A).*

Proof. Our strategy is first to factor τ as $A \to C' \to C$, where C' is an A-algebra such that $mC' \subset H_{C'/A}$, apply Proposition 3.4 to C', and then take the base change of the resulting homomorphism $C' \to D'$ by C. Let f denote the column r-vector with entries f_j, $1 \le j \le r$, and a_i the column r-vector with entries a_{ij}, $1 \le j \le r$, so that (3.6) can be written in the form $f = \sum_{i=1}^{n} a_i u_i$.

Lemma 3.7. *Let A be a ring and $a_1, \dots, a_n$ r-vectors with entries in A. Let Δ denote the ideal generated by all the $r \times r$ minors of the $r \times n$ matrix formed by $a_1, \dots, a_n$ (as usual, we take $\Delta = 0$ if $n < r$). Let J denote the submodule of A^r generated by $a_1, \dots, a_n$. Then $\Delta A^r \subset J$.*

Proof. For an $r \times r$ minor Δ_1 of the matrix $(a_{ij})_{\substack{1 \le i \le r \\ 1 \le j \le n}}$ and an integer q, $1 \le q \le r$, let $b(\Delta_1, q)$ denote the r-vector whose qth entry is Δ_1 and all the other entries 0. By linear algebra, $b(\Delta_1, q) \in J$ for every Δ_1 and q. Since Δ is the ideal generated by all the different minors Δ_1, we have $\Delta A^r \subset J$, as desired. □

We come back to the proof of Proposition 3.6. Since $mA_m \subset \sqrt{\Delta^0_f A_m}$, by Lemma 3.7 there exists $N \gg 0$ such that

$$m^N A^r_m \subset J A^r_m. \tag{3.9}$$

Let $y_1, \dots, y_s$ be a set of generators of m^N. Define r-vectors b_{ij}, $1 \le i \le s$, $1 \le j \le r$, as follows. The only non-zero entry in b_{ij} is the jth one, and that is equal to y_i. Let v_{jk} be new variables and define an A-module M' by

$$M' := \frac{\left(\bigoplus_{i=1}^{n} Au_i\right) \oplus \left(\bigoplus_{\substack{1\le j\le s\\ 1\le k\le r}} Av_{jk}\right)}{\left(\sum_{i=1}^{n} a_i u_i + \sum_{j,k} b_{jk} v_{jk}\right)}. \tag{3.10}$$

We have a surjection $\pi : M' \to M$ defined by sending all the v_{jk} to 0. Let $C' := S_A M'$. The A-module homomorphism π induces an A-algebra homomorphism $C' \to C$ which we also denote by π. Let $\rho' := \rho \circ \pi$. By (3.10), $(y_1, \dots, y_s) \subset H_{C'/A} \cap A$, hence $m = \sqrt{(y_1, \dots, y_s)} \subset H_{C'/A} \cap A$.

(3.10) defines a presentation $0 \to A^r \to A^{n+rs} \to M' \to 0$ which maps to (3.1) in the obvious way. We may identify A^r with L_1. Denoting A^{n+rs} by L_0', we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \xrightarrow{d_1} & L_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\ & & \uparrow \wr & & {\scriptstyle\pi_0}\uparrow & & \uparrow {\scriptstyle\pi} & & \\ 0 & \longrightarrow & L_1 & \xrightarrow{d_1'} & L_0' & \xrightarrow{d_0'} & M' & \longrightarrow & 0 \end{array} \tag{3.11}$$

Since π_0 is surjective, its dual π_0^* is injective. Hence there exists a free finitely generated A-module G such that $\alpha : K \twoheadrightarrow Ker\, d_1^*$ extends to a surjection $K \oplus G \twoheadrightarrow Ker(d_1')^*$. Let $K' := K \oplus G$. We obtain a commutative diagram

$$\begin{array}{ccccc} K & \xrightarrow{\alpha} & L_0^* & \xrightarrow{d_1^*} & L_1^* \\ \downarrow & & {\scriptstyle\pi_0^*}\downarrow & & \downarrow \wr \\ K' & \xrightarrow{\alpha'} & {L_0'}^* & \xrightarrow{{d_1'}^*} & L_1^* \end{array} \tag{3.12}$$

Consider the commutative diagram obtained by tensoring (3.12) with B. Let $u' \in Ker({d_1'}^* \otimes B)$ denote the element corresponding to $\rho'|_{M'} \circ d_1'$. Then $u' = (\pi_0^* \otimes B)(u)$, hence $u' \in (\pi_0^* \otimes B)((\alpha \otimes B)(L_0^*)) \subset (\alpha' \otimes B)(K' \otimes B)$. Thus C' and α' satisfy the hypotheses of Proposition 3.4. Hence there exists a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ \downarrow & & \uparrow {\scriptstyle\psi} \\ C' & \xrightarrow[\phi]{} & D' \end{array}$$

such that D' is a polynomial ring in finitely many variables over A and

$$mD' \subset H_{C'/A} D \subset H_{D'/C'} \tag{3.13}$$

(cf. (3.7)). By construction, $D' = S_A F'$, where $F' := {K'}^*$. Define $D := D' \otimes_{C'} C$. Since smoothness is preserved by base change [10, Chapter 11, (28.E), p. 201], we have

$$H_{D'/C'} D \subset H_{D/C}. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain $mD \subset H_{D/C}$. By (3.9), $\pi_m : M_m' \to M_m$ has a right inverse; in fact, $M_m' \cong M_m \oplus F_1$, where $F_1 \cong A_m^{rs}$. Then $(\pi_0)_m$ has a right inverse, so $(\pi_0^*)_m$ has a left inverse. Hence, $K'_m \cong K_1 \oplus F_1^*$ where K_1 is a free finite A_m-module which surjects onto $(\pi_0^*)_m(Ker(d_1^*)_m)$. Then $F'_m \cong K_1^* \oplus F_1$. Since the map π_m is just the quotient by F_1, $D \otimes_A A_m = \frac{D' \otimes_A A_m}{F_1 D' \otimes_A A_m} = S_{A_m}(K_1^*)$, so $D \otimes_A A_m$ is smooth over A_m. Therefore $H_{D/A} \cap A \not\subset m$, as desired. □

We end this section by proving Proposition 1.6.

Proof of Proposition 1.6. The first step is to reduce to the case when A^* is of *finite type* over A (we prove a slightly more general result for future use).

Lemma 3.8. *Assume that* $A^* = \lim\limits_{\overrightarrow{\alpha}} A_\alpha$, *where* $\{A_\alpha\}$ *is a direct system of finite type* A*-subalgebras, such that, for all* α, A^* *is flat over* A_α *and*

$$H_{A^*/A}B \subset H_{A_\alpha/A}B. \tag{3.15}$$

Then there exists a finite type A*-subalgebra* $A_0^* \subset A^*$, *such that all the homomorphisms in (1.7) are defined over* A_0^*, *so that (1.7) descends to a commutative diagram of finite type* A_0^**-algebras mapping to* B, *and such that conditions (1)–(8) still hold for the descended diagram.*

Proof. Let A_0^* be one of the A_α such that the diagram (1.7), $Ker\ \lambda^*$, the presentation of condition (5) and the presentation (1.4) of M are defined already over A_0^*. Then condition (5) of Proposition 1.6 holds for such an A_0^*. It is clear that conditions (2), (4), (6) and (7) of Proposition 1.6 hold for A_0^* if A_0^* is sufficiently large. Condition (1) holds for A_0^* by (3.15). Moreover, choose A_0^* large enough so that all the presentations used in the calculation of all the Jacobian ideals in conditions (3) and (8), are defined already over A_0^*. Let C_1^* and C_{10}^* be the A_0^*-algebras such that $C^* = C_1^* \otimes_{A_0^*} A^*$ and $C_0^* = C_{10}^* \otimes_{A_0^*} A^*$. Let $C_1^* = \frac{C_{10}^*[V]}{J}$ be a presentation of C_1^* over C_{10}^*, and $g = (g_1, \dots, g_r)$ an r-tuple of elements of J. Since A^* is flat over A_0^*, C^* is flat over C_1^*, and hence $((g) : J)C^* = ((g)C^*) : (JC^*)$. By definition of A_0^*, the ideal Δ_g is the same whether computed in $C_{10}^*[V]$ or in $C_0^*[V]$. Thus $H_{C^*/C_0^*} = H_{C_1^*/C_{10}^*}C^*$, hence condition (8) holds for A_0^*. Condition (3) is proved in exactly the same way. This proves Lemma 3.8. □

Now, since A^* is essentially of finite type over A, we may write $A^* = A'_S$, where A' is of finite type over A and S is a multiplicative subset of A'. Let the direct system $\{A_\alpha\}$ be $\{A'_{S'}\}$, where S' ranges over all the finite subsets of S. For each α, A^* is smooth over A_α; in particular, it is flat. We have $H_{A^*/A} = H_{A_\alpha/A}A^* = H_{A'/A}A^*$, so (3.15) holds. Thus, Lemma 3.8 applies. Replacing A^* by A_0^* of Lemma 3.8, we may assume that A^* is of finite type over A. Now, apply Proposition 3.6 to the A^*-algebra C_0^*. We obtain a commutative diagram

$$\begin{array}{ccc} A^* & \xrightarrow{\sigma} & B \\ \downarrow & & \uparrow \psi_0 \\ C_0^* & \longrightarrow & D_0 \end{array} \tag{3.16}$$

where

$$H_{D_0/A^*} \cap A^* \not\subset m^* \quad \text{and} \tag{3.17}$$

$$m^* D_0 \subset H_{D_0/C_0^*}. \tag{3.18}$$

Put $D^* := D_0 \otimes_{C_0^*} C^*$ and $D := D_0 \otimes_{C_0^*} C_N^*$. We get a commutative diagram

$$\begin{tikzcd} A^* \arrow[r, "\sigma"] \arrow[d] & B & D \arrow[l, "\psi"'] & D^* \arrow[l, "\gamma"'] & D_0 \arrow[l] \\ C \otimes_A A^* \arrow[r, "\pi^*"] & C_N^* \arrow[u, "\rho_N^*"] \arrow[ur] & C^* \arrow[l, "\lambda^*"'] \arrow[ur] & C_0^* \arrow[l, "g^*"'] \arrow[ur] & \end{tikzcd} \tag{3.19}$$

extending (1.7), where ψ is the natural map coming from the tensor product; this gives a diagram (1.2). We now show that this diagram (1.2) satisfies the conclusion of Theorem 1.5 and (1.3). Since $C^*_{P\cap C^*}$ is smooth over $(C_0^*)_{P\cap C_0^*}$ by (1.14), $D^*_{P\cap D^*}$ is smooth over $(D_0^*)_{P\cap D_0^*}$ by base change, and hence also over A^* and A by (1.8), (3.17) and transitivity of smoothness [10, Chapter 11, (28.E), p. 201]. Next, we use (1.11)–(1.13) to show that the natural surjective map $\gamma_P : D^*_{P\cap D^*} \to D_{P\cap D}$ is actually an isomorphism.

Lemma 3.9. *Let $A \to D$ be a local homomorphism of local noetherian rings with D smooth over A. Let m be an ideal of A. Let I be a proper ideal of D such that $I \cap A = (0)$ and*

$$mD \subset \sqrt{(I^2 : I)D}. \tag{3.20}$$

Then $mD \subset \sqrt{Ann_D I}$.

Proof. Since $A \to D$ is local and D is smooth over A, the minimal primes of D are precisely the extensions to D of the minimal primes of A. Let P_0 be a minimal prime of A; then P_0D is a minimal prime of D. We claim that

$$mI \subset P_0D. \tag{3.21}$$

If $m \subset P_0$, there is nothing to prove. Assume that $m \not\subset P_0$; take an element $x \in m \setminus P_0$. The inclusion (3.20) still holds after tensoring over A with $\frac{A_x}{P_0A_x}$; the left hand side of (3.20) becomes the unit ideal, hence so does the right hand side. We obtain

$$I\frac{D}{P_0D} \otimes_A A_x \subset I^2\frac{D}{P_0D} \otimes_A A_x \tag{3.22}$$

in the noetherian ring $\frac{D}{P_0D} \otimes_A A_x$ without zero divisors. Moreover we claim that

$$I\frac{D}{P_0D} \otimes_A A_x \neq \frac{D}{P_0D} \otimes_A A_x. \tag{3.23}$$

Indeed, equality in (3.23) would mean that $x^T = ay + b$, for some $T \in \mathbb{N}$, $a \in P_0$, $y \in D$ and $b \in I$. Let z be an element of A, contained in all the minimal primes of A except P_0. Then za is nilpotent. Then for $S \in \mathbb{N}$ sufficiently large, we have $(zx^T)^S = (azy + zb)^S \in I \cap A \setminus P_0$, which is a contradiction. (3.22) and (3.23) prove that $I\frac{D}{P_0D} \otimes_A A_x = (0)$; by the choice of x, $I \subset P_0D$ and (3.21) is proved. Since this holds for every minimal prime P_0, we obtain that mI is nilpotent. Say, $(mI)^L = (0)$. By (3.20), there is an $N \in \mathbb{N}$ such that $m^N I \subset I^2$. Iterating this $L-1$ times, we obtain $m^{NL} I \subset m^L I^L = (0)$. We have found a power of m annihilating I, as desired. $\square$

We continue with the proof of Proposition 1.6. Extending all the ideals in (1.11)–(1.12) to $D^*_{P\cap D^*}$, we obtain

$$ID^*_{P\cap D^*} \subset (m^*)^N D^*_{P\cap D^*} \tag{3.24}$$

and $m^*D^*_{P\cap D^*}\subset\sqrt{(I^2:I)D^*_{P\cap D^*}}$. By Lemma 3.9, applied to the smooth, local $A^*_{m^*}$-algebra $D^*_{P\cap D^*}$, we have

$$m^*D^*_{P\cap D^*}\subset\sqrt{Ann_{D^*_{P\cap D^*}}ID^*_{P\cap D^*}}. \tag{3.25}$$

In other words, $ID^*_{P\cap D^*}$ is annihilated by some power of m^*, so that

$$ID^*_{P\cap D^*}\subset Ann_{P\cap D^*}(m^*)^\infty D^*_{P\cap D^*}. \tag{3.26}$$

Combining (3.24) and (3.26), we obtain

$$\begin{aligned} Ker\ \gamma_P\equiv ID^*_{P\cap D^*}&\subset (Ann_{D^*_{P\cap D^*}}(m^*)^\infty D^*_{P\cap D^*})\cap((m^*)^N D^*_{P\cap D^*})\\ &=\left((Ann_{A^*_{m^*}}(m^*)^\infty A^*_{m^*})\cap((m^*)^N A^*_{m^*})\right)D^*_{P\cap D^*}, \end{aligned}\tag{3.27}$$

where the last equality holds since $D^*_{P\cap D^*}$ is faithfully flat over $A^*_{m^*}$ (being smooth and local). Now, $A^*_{m^*}$ is smooth over A_m by (1.8), hence fatithfully flat over it. Then all the associated primes of $A^*_{m^*}$ are extended from associated primes of A_m, in particular, are contained in $mA^*_{m^*}$. If $m^*A^*_{m^*}\not\subset mA^*_{m^*}$, then it contains some non-zero divisors, so that $Ann_{A^*_{m^*}}(m^*)^\infty A^*_{m^*}=(0)$ and so $Ker\ \gamma_P=(0)$ by (3.27). Assume that $m^*A^*_{m^*}\subset mA^*_{m^*}$; the opposite inclusion is trivial by definitions, so that $mA^*_{m^*}=m^*A^*_{m^*}$. Hence

$$\begin{aligned} (Ann_{A^*_{m^*}}(m*)^\infty A^*_{m^*})\cap((m*)^N A^*_{m^*})&=(Ann_{A^*_{m^*}}m^\infty A^*_{m^*})\cap m^N A^*_{m^*}\\ &=\left((Ann_{A_m}m^\infty A_m)\cap m^N A_m\right)A^*_{m^*}, \end{aligned}\tag{3.28}$$

where the last equality holds by faithful flatness of $A^*_{m^*}$ over A_m. Now, (1.13), (3.27) and (3.28) prove that $Ker\ \gamma_P=(0)$, so that γ_P is an isomorphism. Since $D_{P\cap D}\cong D^*_{P\cap D^*}$ is smooth over $A^*_{m^*}$ and $A^*_{m^*}$ over A (1.8), $D_{P\cap D}$ is smooth over A. This gives (2) of Theorem 1.5.

Moreover, from (3.18) we get $m^*D\subset H_{D/C^*_N}$ by base change. Combined with (1.8)–(1.10), transitivity of smoothness and the fact that $H_{A/A^*}C\subset H_{C\otimes_A A^*/C}$ (base change), we get

$$\sqrt{H_{C/A}B}\subset P\subset\sqrt{H_{D/C_N}H_{C_N/(C\otimes_A A^*)}H_{A^*/A}B}\subset\sqrt{H_{D/C}B},$$

which gives (1.3). (1) of Theorem 1.5 follows from (1.3) and Property 2.16. □

Remark 3.10. If B is local with maximal ideal P, then, replacing D by D_x, where x is any element of $H_{D/A}\setminus(P\cap D)$, we get a diagram (1.2) with D smooth over A, proving Theorem 1.2.

§4. Smoothing of an isolated, almost complete intersection singularity with no residue field extension

In this section we prove a special case of of Theorem 1.2: Proposition 1.8. Extending the proof from this special case to the general one forms the technical part of the paper and occupies §5–§9. Our main tool in proving Proposition 1.8 will be a transformation of C called **generalized blowing up**. The map π^*, required in the diagram (1.7), will be given as a composition of generalized blowings up. We start by defining generalized blowing up and studying its basic properties. Consider a diagram (1.1), with C of finite type over A. Let m be an ideal of A and $z_1,\dots,z_k$ a set of generators of m. Fix a presentation $C=\frac{A[u]}{I}$. Assume that

$$\rho(C)\subset\sigma(A)+mB \tag{4.1}$$

(note that (4.1) is satisfied in the situation of Proposition 1.8 by the conditions (2) and (4)). We now define the generalized blowing up $\pi : C \to C_1$ of C along m. By (4.1), there exist $c_1, \dots, c_n \in A$ such that $\rho(u_i) - \sigma(c_i) \in mB$. Let v_{ij}, $1 \le i \le n$, $1 \le j \le k$ be independent variables and consider the change of variables

$$u_i - c_i = \sum_{j=1}^{k} z_j v_{ij}, \qquad 1 \le i \le n. \tag{4.2}$$

Write v for $\{v_{ij} \mid 1 \le i \le n, 1 \le j \le k\}$. Equations (4.2) define a homomorphism $\pi_z : A[u] \to A[v]$. Put

$$C_1 := \frac{A[v]}{I_1}, \tag{4.3}$$

where I_1 is the ideal of $A[v]$ generated by the set $\{\pi_z(f) \mid f \in I\}$. The homomorphism π_z induces a homomorphism $C \to C_1$ which by abuse of notation we shall also call π_z. By definition of c_i, there exists a homomorphism $\rho_1 : C_1 \to B$ compatible with σ, τ and ρ.

Definition 4.1. The homomorphism π_z is called a **generalized blowing up** of C along m (with respect to z). We emphasize that the generalized blowing up π_z is a transformation which, given a finite type A-algebra C *together with a fixed presentation* and a set of generators z of m, produces a finite type A-algebra C_1 together with the presentation (4.3).

Remark 4.2. Note that even once we fix a presentation of C and the set z, the map $\rho_1 : C_1 \to B$ is, in general, not unique. In the applications we always pick and fix one such map ρ_1. Note also that we allow the possibility $m = (0)$ in Definition 4.1. In this case, we have $k = 0$ and $z = \emptyset$ and the right hand side of (4.2) is 0.

An important property of generalized blowing up, which follows immediately from (4.2), is that

$$m \subset H_{C_1/C} \cap A; \tag{4.4}$$

this will be used to deduce (3) of Proposition 1.6. The key idea in the proof of Proposition 1.8 (and Theorems 1.2 and 1.5) is to study the behaviour of Jacobian ideals under generalized blowing up. First, we consider the effect of the change of variables (4.2) on an arbitrary column vector of elements of $A[u]$. Let r be a positive integer. Let f be a column r-vector, whose entries are elements of $A[u]$. For $1 \le i \le n$, let $\left(\frac{\partial f}{\partial u_i}\right)$ be the column r-vector with entries in $A[u]$, obtained from f by differentiating every entry with respect to u_i. Write $f = \sum_\alpha a_\alpha u^\alpha$, where α ranges over some finite subset of $\mathbb{N}_0^n$ and $a_\alpha \in A^r$. Let $I(f)$ denote the A-submodule of A^r generated by the a_α. For a submodule J of A^r, we will denote by JB^r the image of J in B^r under σ, and similarly for submodules of C^r. Our main tool will be Taylor's formula:

$$f = f(c) + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}(c)(u_i - c_i) + h, \tag{4.5}$$

where $h \in (u-c)^2 I(f) A[u]^r$. Let $\rho\left(\frac{\partial f}{\partial u_i}\right)$ denote the column r-vector with entries in B, obtained from $\left(\frac{\partial f}{\partial u_i}\right)$ by mapping it to C by the natural map $A[u] \to C$ and then applying ρ to every entry. Let $J(f)$ denote the submodule of B^r generated by the r-vectors $\rho\left(\frac{\partial f}{\partial u_i}\right)$, $1 \le i \le n$. We have

$$J(f) \subset I(f)B^r. \tag{4.6}$$

Consider the generalized blowing up $\pi : C \to C_1$ given by (4.2). The key point is to compare $I(f)$ with $I(\pi(f))$ and $J(f)$ with $J(\pi(f))$.

Lemma 4.3. *Assume that the entries of the r-vector f belong to I. Then:*

$$J(\pi(f)) = mJ(f), \tag{4.7}$$

$$I(\pi(f))B^r \subset mJ(f) + m^2 I(f)B^r. \tag{4.8}$$

Proof. (4.7) follows from (4.2) by the chain rule. To prove (4.8), substitute (4.2) in (4.5). We obtain

$$\pi(f) = f(c) + \sum_{i,j} \frac{\partial f}{\partial u_i}(c) z_j v_{ij} + h_1, \tag{4.9}$$

where $h_1 \in m^2(v)^2 I(f)A[v]^r$. Let $J_0(f)$ denote the submodule of A^r generated by $\frac{\partial f}{\partial u_1}(c), \dots, \frac{\partial f}{\partial u_n}(c)$. Since for each i we have $\frac{\partial f}{\partial u_i} - \frac{\partial f}{\partial u_i}(c) \in (u-c)I(f)A[u]^r$, we have

$$J_0(f)B^r \subset J(f) + mI(f)B^r. \tag{4.10}$$

Applying ρ to (4.9), identifying $f(c)$ with its image in B and using that f has entries in I (and hence maps to 0 in B^r), we obtain

$$f(c) \in (mJ_0(f) + m^2 I(f))B^r. \tag{4.11}$$

By (4.9)–(4.11), $I(\pi(f))B^r \subset (mJ_0(f) + m^2 I(f))B^r = mJ(f) + m^2 I(f)B^r$. Lemma 4.3 is proved. □

Now consider a sequence

$$C \xrightarrow{\pi_1} C_1 \xrightarrow{\pi_2} \dots \xrightarrow{\pi_N} C_N \tag{4.12}$$

of N generalized blowings up along m. For each i, we have a homomorphism $\rho_i : C_i \to B$; the ρ_i commute with the π_i in (4.12). Here we are assuming that

$$\rho_i(C_i) \subset \sigma(A) + mB \tag{4.13}$$

for each $i < N$. Note that this assumption holds for *any* algebra C_i if $mB = P$ and (2) of Proposition 1.8 is satisfied. In the next lemma, assume that $C = \frac{A[u]}{I}$ is an almost complete intersection at $P \cap C$. Let f be the column r-vector with entries f_i, and let $J(f)$ be as above. By (1.17) and Lemma 3.7, we have

$$m^N B^r \subset J(f) \tag{4.14}$$

for all N sufficiently large. Since A is assumed to be noetherian and since the $Ann_A m^i$ form an ascending chain of ideals, $Ann_A m^\infty = Ann_A m^i$ for some i. Using the Artin–Rees lemma, we obtain, for all N sufficiently large,

$$m^N \cap (Ann_A m^\infty) = m^N \cap (Ann_A m^i) \subset m^i Ann_A m^i = (0). \tag{4.15}$$

Lemma 4.4. *Consider a diagram (1.1). Assume that A is local with maximal ideal m, that (1) and (3) of Proposition 1.8 hold, that $mB = P$ and that (4.13) holds for all the A-algebras C_i appearing in (4.12), so that the sequence (4.12) is well defined (note that the last condition holds automatically if we have (2) of Proposition 1.8). Take $N \in \mathbb{N}$ in (4.12) to be sufficiently large so that both (4.14) and (4.15) hold (again, if $r = 0$, we regard (4.14) as being vacuously true). Let $\pi^* := \pi_N \circ \cdots \circ \pi_1$. Then the map $\pi^* : C \to C_N$ fits in a commutative diagram (1.7) (with $A = A^*$ and $m = m^*$), satisfying (1)–(5) and (8) of Proposition 1.6. If, in addition,*

$$m^{2N} \subset I(f^{(N)}), \tag{4.16}$$

then (7) of Proposition 1.6 holds.

Proof. First, we construct the diagram (1.7). By definition of generalized blowing up, each C_i comes together with a specific presentation (cf. (4.3)); let $C_N = \frac{A[u^{(N)}]}{I_N}$ be the given presentation of C_N. Let $f_i^{(N)} := (\pi_N \circ \cdots \circ \pi_2 \circ \pi_1)(f_i)$ for $1 \leq i \leq r$. By Lemma 4.3, (4.14) and induction on N, we have $I(f^{(N)})B^r \subset m^N J(f) + m^{2N} I(f) B^r = m^N J(f) = J(f^{(N)})$, so

$$I(f^{(N)})B^r = J(f^{(N)}) \tag{4.17}$$

by (4.6). Let $(f^{(N)})$ denote the ideal of $A[u^{(N)}]$, generated by $f_1^{(N)}, \ldots, f_r^{(N)}$. Put $C^* := \frac{A[u^{(N)}]}{(f^{(N)})}$. We have natural homomorphisms $\lambda^* : C^* \to C_N$ and $\rho^* : C^* \to B$, given by $\rho^* = \rho_N \circ \lambda^*$; note that $Ker\ \lambda^* = I_N C^*$. Let $(a_1, \ldots, a_l)$ denote a minimal set of generators of $I(f^{(N)})$. Write $f^{(N)} = \sum_{i=1}^{l} a_i g_i$, where $g_i \in A[u^{(N)}]$. Let $G_1, \ldots, G_l$ be new variables. Let $F_1, \ldots, F_r$ denote the entries of the r-vector $\sum_{i=1}^{l} a_i G_i$. We will write (F) for $(F_1, \ldots, F_r)$. Let $C_0^* := \frac{A[G_1, \ldots, G_l]}{(F)}$. Let $g : C_0^* \to C^*$ be the map which sends G_i to g_i. This completes the construction of the diagram (1.7). Again, note that we allow the possibility $r = l = 0$, $(f^{(N)}) = (F) = (0)$. Next, we show that the diagram (1.7) thus constructed satisfies (1)–(5) and (8) of Proposition 1.6. (1) and (2) of Proposition 1.6 are trivial. (1.10) follows immediately from (4.4) and Property 2.16 by induction on N. From (1.18) we obtain $mC^*_{P \cap C^*} \subset \sqrt{((I_N^2) : I_N) C^*_{P \cap C^*}}$; this gives (1.12). (1.13) is nothing but (4.15). (5) of Proposition 1.6 is true by definition. It remains to prove (8).

Proof of (8). Let $K = \frac{B}{P}$. Let $\frac{\partial g}{\partial u^{(N)}}$ denote the $l \times n$ matrix whose ijth entry is $\frac{\partial g_i}{\partial u_j^{(N)}}$. Since $J(f^{(N)})$ is generated by $\rho_N \left(\frac{\partial f^{(N)}}{\partial u_i^{(N)}} \right) = \sum_{j=1}^{l} a_j \rho_N \left(\frac{\partial g_j}{\partial u_i^{(N)}} \right)$, $1 \leq i \leq n$, and since, by (4.17) and Nakayama's lemma, $(a_1, \ldots, a_l)$ induces a *minimal* set of generators of the K-vector space $\frac{J(f^{(N)})}{PJ(f^{(N)})}$, we have

$$\operatorname{rk} \left(\frac{\partial g}{\partial u^{(N)}} \right) = l \qquad \bmod \left(P \cap A[u^{(N)}] \right). \tag{4.18}$$

Since C^* is defined over C_0^* by the equations $g_i = G_i$, $1 \leq i \leq l$, (4.18) implies that $C^*_{P \cap C^*}$ is smooth over C_0^*, that is, $H_{C^*/C_0^*} \not\subset P \cap C^*$, as desired. Again, the above is trivially true if $r = 0$, for then $C^* = C_0^* = A[u^{(N)}]$. $\square$

Finally, suppose (4.16) holds. By definitions, Δ_F^0 is the ideal generated by all the $r \times r$ minors of the $r \times l$ matrix formed by $a_1, \dots, a_l$, so (4.16) implies that $m \subset \sqrt{\Delta_F^0}$, as desired. This completes the proof of Lemma 4.4. $\square$

Proof of Proposition 1.8. Put $A^* = A_m$. From now on, to simplify the notation, we will replace A by A^*, C by $C \otimes_A A^*$, and assume that A is local with maximal ideal m (in particular, σ is faithfully flat). Under this assumption, we will construct a diagram (1.7) with $A = A^*$. Take N as in Lemma 4.4 and consider the sequence (4.12) of N generalized blowings up. By Lemma 4.4, (1)–(5) and (8) of Proposition 1.6 are satisfied. Proposition 1.6 (6) holds by Remark 3.1, since B is flat over A. It remains to prove (4.16) to infer (7). $\square$

Lemma 4.5. *If B is flat over A, then (4.16) holds.*

Proof. We have $m^{2N}B^r \subset J(f^{(N)})B^r$ by (4.7), (4.14), and induction on N. By (4.17), this gives $m^{2N}B^r \subset I(f^{(N)})B^r$. (4.16) follows by faithful flatness of B over A. This completes the proof of Proposition 1.8. $\square$

Remark 4.6. Suppose that assumption (4) in Proposition 1.8 is replaced by saying that $\sigma : A \to B$ is formally smooth in the P-adic topology (by Proposition 2.7, this is weaker than being a regular homomorphism). Then the conclusion of Proposition 1.8 still holds: we have only to reduce to the situation when $mB = P$. This can be done as follows. Let $K := \frac{B}{P} \cong \frac{A_m}{m}$ and $B_0 := \frac{B}{mB}$. The map $K \to B_0$ induced by σ is formally smooth in the PB_0-adic topology (by base change: [10, Chapter 11, (28.E), p. 201]), so B_0 is a regular local ring (Proposition 2.6). Let $x_1, \dots, x_d$ be elements of B which induce a regular system of parameters of B_0 and let $X_1, \dots, X_d$ be independent variables. Write X for $X_1, \dots, X_d$. Consider the map $\sigma_X : A[X]_{P \cap X} \to B$ which sends X_i to x_i. The map σ_X is flat by Corollary 2.22. Now we can apply Proposition 1.8 to σ_X. This gives diagram (1.7) with $A^* = A[X]_{P \cap A[X]}$. Since $A[X]_m$ is smooth over A, (1.8) is satisfied and we are done.

§5. Smoothing of an isolated singularity over a local ring with no residue field extension

Consider a diagram (1.1). Let P be a minimal prime of $H_{C/A}B$ and let $m := P \cap A$. In this section we prove

Proposition 5.1. *Assume that (1), (2) and (4) of Proposition 1.8 hold. Then there exists a diagram (1.7) satisfying conditions (1)–(8) of Proposition 1.6 (and hence the conclusion of Theorem 1.2 holds in this case).*

With a view to §§8–9, we will start out working under more general hypotheses than those of Proposition 5.1, and gradually impose more restrictions on our diagram (1.1) as we go along. The idea is to show that ρ factors through a map $C \to C_L$ such that $P \subset \sqrt{H_{C_L/C}B}$ and C_L is an almost complete intersection over A at $P \cap C_L$. Once this is done, we will invoke Proposition 1.8 and the proof will be complete. We start with any diagram (1.1) whatsoever of noetherian rings. Let $C = \frac{A[u_1, \dots, u_n]}{I}$ be a presentation of C. Let $\frac{\mathcal{I}}{\mathcal{I}^2}$ denote the coherent sheaf on Spec C such that $\Gamma\left(\text{Spec } C, \frac{\mathcal{I}}{\mathcal{I}^2}\right) = \frac{I}{I^2}$ (in what follows we will adopt the following convention: ideals and modules will be denoted by capital letters, and their sheafications by script capital letters). Restricted to the smooth locus of C over A, the sheaf $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\operatorname{Spec} C\setminus V(H_{C/A})}$ is nothing but the conormal bundle of $\operatorname{Spec} C \setminus V(H_{C/A})$ in Spec $A[u]$. The first step of the proof is to achieve the situation when the vector bundle $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\operatorname{Spec} C\setminus V(H_{C/A})}$ is trivial. This is given by Elkik's lemma, which we now invoke. Let $C' := S_C(\frac{I}{I^2})$. Extend ρ to C' by setting it to be (for example) the 0 map on the positive degree part of $S_C(\frac{I}{I^2})$.

Lemma 5.2 (Elkik [7, Lemma 3]). *There exists a presentation* $C' = \frac{A[u']}{I'}$ *such that the restriction* $\frac{\mathcal{I}'}{\mathcal{I}'^2}\Big|_{\operatorname{Spec} C'\setminus V(H_{C'/A})}$ *is the trivial vector bundle.*

Since $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\operatorname{Spec} C\setminus V(H_{C/A})}$ is *locally* free, for any prime ideal $Q \subset C'$, C'_Q is smooth over C whenever $C_{Q\cap C}$ is smooth over A. Therefore

$$H_{C/A}C' \subset H_{C'/C} \tag{5.1}$$

(this will be needed to prove (1.10)). Let P be as in Theorem 1.5 and let $m = P\cap A$. If $H_{C'/A}B \not\subset P$, we may take $C_L = D = C'$ and there is nothing more to prove. If $H_{C'/A}B \subset P$, then P is a minimal prime of $H_{C'/A}B$ by Property 2.16 and (5.1). In this case, replace C by C'. From now on, assume that the vector bundle $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\operatorname{Spec} C\setminus V(H_{C/A})}$ is trivial. If $V(mH_{C/A}C_{P\cap C}) \neq \operatorname{Spec}\ C_{P\cap C}$, let $r := \operatorname{rk} \frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\operatorname{Spec} C\setminus V(H_{C/A})}$ and let $\bar{f}_1,\dots,\bar{f}_r$ be global sections of $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\operatorname{Spec} C\setminus V(H_{C/A})}$, which generate $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\operatorname{Spec} C\setminus V(H_{C/A})}$. If $V(mH_{C/A}C_{P\cap C}) = \operatorname{Spec}\ C_{P\cap C}$, set $r = 0$ and let $\{\bar{f}_1,\dots,\bar{f}_r\}$ be the empty set. Let $z_1,\dots,z_k$ be a set of generators of m. To achieve the situation when C is an almost complete intersection at $P\cap C$, we iterate the generalized blowing up π_z (see Definition 4.1). Consider a sequence

$$C \xrightarrow{\pi_1} C_1 \xrightarrow{\pi_2} \dots \xrightarrow{\pi_L} C_L \xrightarrow{\pi_{L+1}} \cdots \tag{5.2}$$

of generalized blowings up (we are assuming that (4.13) holds for each i in (5.2), so that (5.2) is well defined). Let $C_i = \frac{A[u_1^{(i)},\dots,u_{n_i}^{(i)}]}{I_i}$ be the presentation of C_i obtained, recursively, from the definition of generalized blowing-up (here $n_{i+1} = kn_i$; cf. (4.2) and (4.3)). The purpose of the next several lemmas is to show that, under some additional hypotheses, after a finite number L of such blowings up we can ensure that there exist $f_1,\dots,f_r \in I_L$ such that for $1 \le i \le r$, $\bar{f}_i$ is the natural image of f_i, up to multiplication by an element of $A[u^{(L)}] \setminus (P\cap A[u^{(L)}])$. This will mean that C_L is an almost complete intersection over A at $P\cap C$ and our proof will be finished.

Lemma 5.3. *For all $i \in \mathbb{N}$, the vector bundle* $\frac{\mathcal{I}_i}{\mathcal{I}_i^2}\Big|_{\operatorname{Spec}\ C_i\setminus V(mH_{C/A}C_i)}$ *is trivial.*

Proof. By definition,

$$\frac{I_i}{I_i^2} \cong \frac{I_{i-1}A[u^{(i)}]}{I_{i-1}^2A[u^{(i)}]}, \quad i \in \mathbb{N}. \tag{5.3}$$

Since $A[u^{(i)}]$ is flat over $A[u^{(i-1)}]$ away from $V(mA[u^{(i-1)}])$, away from that locus (5.3) can be rewritten as

$$\frac{\mathcal{I}_i}{\mathcal{I}_i^2}\Big|_{\operatorname{Spec}\ C_i\setminus V(mC_i)} \cong \pi_i^*\left(\frac{\mathcal{I}_{i-1}}{\mathcal{I}_{i-1}^2}\right)\Big|_{\operatorname{Spec}\ C_i\setminus V(mC_i)}, \quad i \in \mathbb{N}, \tag{5.4}$$

and the lemma follows immediately by induction on i. □

For each i, consider the exact sequence $\frac{I_i}{I_i^2} \xrightarrow{d_i} C_i^{n_i} \xrightarrow{\omega_i} \Omega_{C_i/A}$ of C_i-modules and the corresponding exact sequence

$$\frac{\mathcal{I}_i}{\mathcal{I}_i^2} \xrightarrow{\tilde{d}_i} \mathcal{O}^{n_i}_{\text{Spec } C_i} \xrightarrow{\tilde{\omega}_i} \Omega^1_{\text{Spec } C_i/A} \tag{5.5}$$

of sheaves of $\mathcal{O}_{\text{Spec } C_i}$-modules. First, let $i = 0$ in (5.5). Since $\bar{f}_1, \dots, \bar{f}_r \in \Gamma(\text{Spec } C \setminus V(H_{C/A}), \frac{\mathcal{I}}{\mathcal{I}^2})$, we have $\tilde{d}_0(\bar{f}_j) \in \Gamma(\text{Spec } C \setminus V(H_{C/A}), \mathcal{O}^n_{\text{Spec } C})$, $1 \le j \le r$, and hence

$$H^{L_1}_{C/A}\tilde{d}_0(\bar{f}_j) \in \Gamma\left(\text{Spec } C, \mathcal{O}^n_{\text{Spec } C}\right) \equiv C^n, \ 1 \le j \le r, \text{ for all } L_1 \gg 0. \tag{5.6}$$

Lemma 5.4. *Assume there is a positive integer L_0 such that*

$$m^{L_0}(C_{L_0})_{P\cap C_{L_0}} \subset mH_{C/A}(C_{L_0})_{P\cap C_{L_0}}. \tag{5.7}$$

Let L_1 be such that (5.6) holds and let $L = L_0L_1$. Then C_L is an almost complete intersection over A at $P \cap C_L$.

Proof. We have

$$m(C_L)_{P\cap C_L} \subset (H_{C_L/A})_{P\cap C_L} \tag{5.8}$$

by Property 2.16, induction on L and (5.7). Combining Lemma 5.3, (5.4) and (5.7), we see that the sections

$$(\pi_L \circ \dots \circ \pi_1)^*\bar{f}_1, \dots, (\pi_L \circ \dots \circ \pi_1)^*\bar{f}_r \in \Gamma\left(\text{Spec}(C_L)_{P\cap C_L} \setminus V(m(C_L)_{P\cap C_L}), \frac{\mathcal{I}_L}{\mathcal{I}_L^2}\right)$$

freely generate $\left.\frac{\mathcal{I}_L}{\mathcal{I}_L^2}\right|_{\text{Spec}(C_L)_{P\cap C_L}\setminus V(m(C_L)_{P\cap C_L})}$. It remains to show (3) of Definition 1.7. This is given by the following lemma.

Lemma 5.5. *There exist elements $f_1^{(L)}, \dots, f_r^{(L)} \in I_L$ whose natural images in $\Gamma(\text{Spec}(C_L)_{P\cap C_L} \setminus V(m(C_L)_{P\cap C_L}), \frac{\mathcal{I}_L}{\mathcal{I}_L^2})$ are $(\pi_L \circ \dots \circ \pi_1)^*\bar{f}_1, \dots, (\pi_L \circ \dots \circ \pi_1)^*\bar{f}_r$, respectively, up to multiplication by an element of $C_L \setminus P \cap C_L$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
\frac{\mathcal{I}}{\mathcal{I}^2} & \xrightarrow{\tilde{d}_0} & \mathcal{O}^n_{\text{Spec } C} & \xrightarrow{\tilde{\omega}_0} & \Omega^1_{\text{Spec } C/A} \\
\downarrow & & \mathbf{z}_i\downarrow & & \downarrow \\
\frac{\mathcal{I}_i}{\mathcal{I}_i^2} & \xrightarrow{\tilde{d}_i} & \mathcal{O}^{n_i}_{\text{Spec } C_i} & \xrightarrow{\tilde{\omega}_i} & \Omega^1_{\text{Spec } C_i/A}
\end{array} \tag{5.9}$$

induced by the map $A[u] \to A[u^{(i)}]$, given by iterating (4.2). First, let $i = 1$ in (5.9). By (4.2) and the chain rule, $\mathbf{z}_1$ sends an n-vector with entries $b_1, \dots, b_n \in \Gamma(U, \mathcal{O}^n_{\text{Spec } C})$ (where U is an open set of Spec C) to an (nk)-vector with entries z_jb_l, $1 \le j \le k$, $1 \le l \le n$. Next, let $i = L$ in (5.9). By induction on L, $\mathbf{z}_L$ sends $(b_1, \dots, b_n)$ to an n_L-vector all of whose components are of the form $z^\alpha b_l$, $1 \le l \le n$, where $|\alpha| = L$. Combining this with (5.6)–(5.7) we obtain, after localization at $P \cap C_L$, that $\tilde{d}_L((\pi_L \circ \dots \circ \pi_1)^*\bar{f}_j)$ extends to an element of

$$\Gamma(\text{Spec}(C_L)_{P\cap C_L}, \mathcal{O}^{n_L}_{\text{Spec}(C_L)_{P\cap C_L}}) \equiv (C_L)^{n_L}_{P\cap C_L} \tag{5.10}$$

for $1 \le j \le r$. Since $\tilde{d}_L((\pi_L \circ \dots \circ \pi_1)^*\bar{f}_j) \in Ker\,\tilde{\omega}_L$, by (5.9) there exist $\bar{f}_1^{(L)}, \dots, \bar{f}_r^{(L)} \in \frac{I_L}{I_L^2} \otimes_{C_L} (C_L)_{P\cap C_L}$ such that $d_L(\bar{f}_j^{(L)}) = \tilde{d}_L((\pi_L \circ \dots \circ \pi_1)^*\bar{f}_j)$,

$1 \leq j \leq r$, under the identification (5.10). Since $\tilde{d}_L$ is injective away from the non-smooth locus $V(H_{C_L/A}(C_L)_{P\cap C_L})$ of $(C_L)_{P\cap C_L}$, by (5.8) it is injective away from $V(m(C_L)_{P\cap C_L})$. Since the $(\pi_L\circ\cdots\circ\pi_1)^*\bar{f}_j$ generate $\frac{\mathcal{I}_L}{\mathcal{I}_L^2}\Big|_{\mathrm{Spec}(C_L)_{P\cap C_L}\setminus V(m(C_L)_{P\cap C_L})}$, the elements $\bar{f}_j^{(L)}$ generate $\frac{\mathcal{I}_L}{\mathcal{I}_L^2}\otimes_{C_L}(C_L)_{P\cap C_L}$ away from $V(m(C_L)_{P\cap C_L})$. Since Spec $A[u^{(L)}]_{P\cap A[u^{(L)}]}$ is affine, $H^1(\mathrm{Spec}\ A[u^{(L)}]_{P\cap A[u^{(L)}]},\mathcal{I}_L^2)=0$, so $\bar{f}_1^{(L)}, \dots, \bar{f}_r^{(L)}$ can be lifted to $f_1^{(L)},\dots,f_r^{(L)}\in(I_L)_{P\cap A[u^{(L)}]}\equiv\Gamma(\mathrm{Spec}\ A[u^{(L)}]_{P\cap A[u^{(L)}]},\mathcal{I}_L)$. Multiplying $f_1^{(L)},\dots,f_r^{(L)}$ by an element of $A[u^{(L)}]\setminus P\cap A[u^{(L)}]$, we may take $f_1^{(L)},\dots,f_r^{(L)}\in I_L$. This proves Lemma 5.5 and hence also Lemma 5.4. □

Proof of Proposition 5.1. We want to apply Lemma 5.4. For that, we must show that there exists L_0 such that (5.7) holds. Since $P=\sqrt{H_{C/A}B}$, there exists L_0 such that

$$m^{L_0}B\equiv P^{L_0}B\subset H_{C/A}PB\equiv mH_{C/A}B. \tag{5.11}$$

We claim that (5.7) holds for this L_0. Indeed, let $h_1,\dots,h_t$ be a set of generators of $H_{C/A}$. Let $h_j^{(i)}$ denote the image of h_j in C_i. By (4.2), (4.5) and induction on i, $h_j^{(L)}$ can be written as

$$h_j^{(L)}=g_j^{(L)}+q_j^{(L)}, \tag{5.12}$$

where $g_j^{(L)}\in A$ and $q_j^{(L)}\in m^LC_L$. Let $H'_L:=(g_1^{(L)},\dots,g_t^{(L)})A$.

Lemma 5.6. *Let Q be a module over a noetherian ring C and M an ideal of C with $M\subset Jac(C)$. Let H and H' be submodules of Q, with sets of generators $H=(h_1,\dots,h_t)$ and $H'=(g_1,\dots,g_t)$. Assume that $g_j-h_j\in M^LQ$, $1\leq j\leq t$, and $M^LQ\subset MH'$. Then $H=H'$.*

Proof. Since $h_j\in H'+M^LQ$, $1\leq j\leq t$, we have $H\subset(H'+M^LQ)=H'$ and $H'\subset H+M^LQ\subset H+MH'$. The result follows by Nakayama's lemma. □

By (5.11)–(5.12) and Lemma 5.6, $H_{C/A}B=H'_{L_0}B$. Then (5.11) implies that $m^{L_0}B\subset mH'_{L_0}B$, hence $m^{L_0}\subset mH'_{L_0}$ by faithful flatness of σ, hence $m^{L_0}C_{L_0}\subset mH'_{L_0}C_{L_0}$. (5.7) follows from (5.12) and Lemma 5.6. Thus we may apply Lemma 5.4. By Lemma 5.4, C_L is an almost complete intersection over A at $P\cap C_L$. Apply Proposition 1.8 with C replaced by C_L. We construct a diagram (1.7) (with $A^*=A_m$). (1.10) follows from (5.1), (4.4) and Property 2.16. (1)–(2) and (4)–(8) of Proposition 1.6 follow immediately by Proposition 1.8 and we are done. □

§6. Separability in field extensions

Let $\sigma:(A,m,k)\to(B,P,K)$ be a regular homomorphism of local noetherian rings. One of the difficulties in proving Theorem 1.2 comes from the fact that the residue field extension $k\to K$ induced by σ need not be separable. However, as we shall see in §7, we always have $\dim_K H_1(k,K,K)\leq\dim B-\dim A<\infty$. In this section, preliminary to §7, we study field extensions with $\dim H_1(k,K,K)<\infty$.

Notation. If $A \to B$ is a ring extension with char $A = p > 0$, AB^p will denote the A-subalgebra of B, generated by the set B^p. Of course, if A and B are fields, then AB^p is a subfield of B since, in that case, if $a \in AB^p \setminus \{0\}$, then $a^{-1} = \left(a^{-1}\right)^p a^{p-1} \in AB^p$.

We start with a general observation about Kähler differentials in field extensions of positive characteristic. Let $k \to K$ be a field extension.

Proposition 6.1. *Assume that* char $k = p > 0$. *Let $a \in K$; consider $da \in \Omega_{K/k}$. We have $da = 0 \iff a \in kK^p$.*

Proof. $\Longleftarrow$ is immediate.

$\Longrightarrow$ It is sufficient to consider the case when K is finitely generated over k. From the Jacobi–Zariski sequence for the triple $k \to kK^p \to K$ and from the fact that $\Omega_{kK^p/k} \otimes_{kK^p} K \to \Omega_{K/k}$ is the zero map, we get the isomorphism $\Omega_{K/kK^p} \tilde{\to} \Omega_{K/k}$. Thus, replacing k by kK^p does not change the problem, that is, we may assume, in addition, that $K^p \subset k$. Then K can be written as $K = \frac{k[x_1,\dots,x_n]}{(x_1^p - a_1,\dots,x_n^p - a_n)}$, $a_i \in k$. Moreover, if $a \notin k$, then we may choose $x_1 = a$ and therefore $da \neq 0$. □

Corollary 6.2. *We have $\Omega_{K/k} = 0$ if and only if $K = kK^p$. More generally, consider a subset $u_\Psi = \{u_\psi \mid \psi \in \Psi\} \subset K$. The elements du_ψ, $\psi \in \Psi$ form a K-basis of $\Omega_{K/k}$ if and only if u_Ψ is a minimal set of generators of K over kK^p.*

A set u_Ψ satisfying the equivalent conditions of Corollary 6.2 is called a p-**basis** of K.

Remark 6.3. Let $\delta : k \to K$ be a finitely generated extension of fields of characteristic $p > 0$. A p-basis of K over k can be constructed as follows. Decompose δ as $k \to K_t \to K_s \to K$, where K_t is purely transcendental over k, K_s is separable algebraic over K_t and K is algebraic and purely inseparable over K_s. Moreover, choose this decomposition in such a way as to minimize $\dim_K \Omega_{K/K_t}$. Let u_Λ be a minimal set of generators of K_t over k and v_Φ a p-basis of K over K_s. Then it is easy to see that $u_\Lambda \cup v_\Phi$ forms a p-basis of K over k (indeed, $du_\Lambda \cup dv_\Phi$ generate $\Omega_{K/k}$ by definition; moreover, they are linearly independent: a non-trivial K-linear dependence relation among $du_\Lambda \cup dv_\Phi$ would imply that one of the w_λ, $\lambda \in \Lambda$ can be removed and replaced by one of the v_ϕ, which contradicts the minimality assumption on $\dim_K \Omega_{K/K_t}$). In particular, $\Omega_{K/k} = 0$ if and only if K is separable algebraic over k. If K is not finitely generated over k, then the extension $k \hookrightarrow k\left(t^{p^{-\infty}}\right) \equiv k\left(t, t^{\frac{1}{p}}, t^{\frac{1}{p^2}}, \dots\right)$, where t is transcendental over k, provides a counterexample to all these statements.

Let $\delta : k \to K$ be any field extension (no assumptions on the characteristic). Let $\{w_\lambda\}_{\lambda \in \Lambda}$ be a maximal family of elements of K such that

(1) $\{w_\lambda\}_{\lambda \in \Lambda}$ are algebraically independent over k.
(2) $\{dw_\lambda\}_{\lambda \in \Lambda}$ are linearly independent over K in $\Omega_{K/k}$.

Write $\{w_\Lambda\}$ for $\{w_\lambda\}_{\lambda \in \Lambda}$. Let $\{v_\phi\}_{\phi \in \Phi}$ be a set of elements of K such that $\{dw_\Lambda\} \cup \{dv_\Phi\}$ form a basis for the K-vector space $\Omega_{K/k}$. Let $K_\Lambda := k(w_\Lambda)$ and let K_Φ denote the subfield of K generated by v_Φ over K_Λ. We get a decomposition of the extension $k \to K$:

$$k \xrightarrow{\alpha} K_\Lambda \xrightarrow{\beta} K_\Phi \xrightarrow{\gamma} K, \tag{6.1}$$

where α is purely transcendental, β is an inseparable algebraic extension and γ is unramified (i.e. $\Omega_{K/K_\Phi} = 0$).

Remark 6.4. In the case when K is finitely generated over K_Λ (in particular, whenever K is finitely generated over k), we have some extra information about the size of Φ and Ω_{K/K_Λ}. First of all, in this case we have $\#\Phi < \infty$. Secondly, K must be algebraic over K_Λ, otherwise we could enlarge the set Λ (cf. Remark 6.3), and this would contradict the fact that w_Λ is the maximal set satisfying conditions (1) and (2). If K is finitely generated over K_Φ, then K is separable algebraic over K_Φ, since $\Omega_{K/K_\Phi} = 0$ (cf. Remark 6.3).

Next, we prove a generalization of the primitive element theorem, which deals with the minimal number of generators of a finitely generated field extension.

Theorem 6.5. *Let $\delta : k \to K$ be a finitely generated field extension. Let $d := \dim_K \Omega_{K/k}$. Then the minimal number of generators of K over k is equal to $d+1$ if δ is separable but not pure transcendental, and to d otherwise.*

Proof. Consider a decomposition (6.1) of δ. Let $t := \operatorname{tr.deg} K/k$, so that $t = \#\Lambda$ and $d = t + \#\Phi$ (cf. Remark 6.3). First, suppose δ is separable. In this case $t = d$. Then the minimal number of generators of K is at least d. If K can be generated by exactly d elements, then it is pure transcendental over k. Otherwise K is generated by one element over K_Λ by the primitive element theorem, hence the minimal number of generators of K over k is $d+1$. This proves the theorem in the separable case.

Suppose δ is not separable. It is obvious that K cannot be generated by fewer than d elements. Let us prove that d elements are enough. Since K_Λ is generated over k by t elements and $\dim_K \Omega_{K/K_\Lambda} = d - t$, we may replace k by K_Λ. In other words, we may assume that K is algebraic over k. Our proof is by induction on d. First, let $d = 1$. Let k_s be the separable closure of k in K and let v be any element of K such that dv generates $\Omega_{K/k} \cong \Omega_{K/k_s}$. Then $K = k_s(v)K^p$ by Corollary 6.2, hence $K = k_s(v)K^{p^n}$ for all n, so that $K = k_s(v)$. Now, it is well known and easy to prove that a composition of a separable algebraic extension with a simple algebraic extension is again simple [9, §VII.6, Theorem 14, p. 185 and Exercise 4, p. 190]. The case $d = 1$ is proved.

Next, let $d > 1$. Let $v_1, \dots, v_d$ be a set of elements of K which induce a basis of $\Omega_{K/k}$. We have $K = k_s(v_1, \dots, v_d)K^p$, hence $K = k_s(v_1, \dots, v_d)K^{p^n}$ for all n, hence $K = k_s(v_1, \dots, v_d)$. Now, $k_s(v_1)$ is a simple extension of k by the $d = 1$ case, and $K = k_s(v_1, \dots, v_d)$ is generated over $k_s(v_1)$ by $d-1$ elements, hence K is generated over k by d elements, as desired. $\square$

Remark 6.6. Another way to phrase Theorem 6.5 is that if $\sigma : k \to K$ is an inseparable finitely generated field extension, then the set v_Φ of (6.1) can always be chosen in such a way that $K = K_\Phi$. Indeed, if σ is inseparable and finitely generated, Theorem 6.5 says that the smallest number of generators of K over K_Λ is $\#\Phi$, hence we may choose v_Φ so that $K = K_\Phi \equiv K_\Lambda(v_\Phi)$.

Let V_ϕ, $\phi \in \Phi$, be independent variables. Let I denote the kernel of the map $K_\Lambda[V_\Phi] \to K_\Phi$ which sends V_ϕ to v_ϕ. Choose a well-ordering of Φ. For an element $\phi \in \Phi$, define $\Phi_\phi := \{\phi' \in \Phi \mid \phi' < \phi\}$. Let K_ϕ denote the subfield of K_Φ generated by v_{Φ_ϕ} over K_Λ. Let $g_\phi \in K_\phi[V_\phi]$ denote the (monic) minimal polynomial of v_ϕ over K_ϕ. Pick and fix a representative G_ϕ of g_ϕ in $K_\Lambda[V_{\Phi_\phi}][V_\phi]$. Since $\frac{K_\Lambda[V_\Phi]}{(G_\Phi)}$ is a field, the relations G_Φ form a set of generators of I. By construction, the elements G_Φ

form a minimal set of generators of I, hence they induce a basis of the K_Φ-vector space $\frac{I}{I^2}$.

Lemma 6.7. *We have a natural isomorphism* $H_1(k, K_\Phi, K_\Phi) \cong \frac{I}{I^2}$. *In particular, the set* Φ *has the same cardinality as any basis of the* K_Φ*-vector space* $H_1(k, K_\Phi, K_\Phi)$.

Proof. By definition, the elements $\{dw_\Lambda\} \cup \{dv_\Phi\}$ are K-linearly independent in $\Omega_{K/k}$. Since there is a natural homomorphism $\Omega_{K_\Phi} \otimes_{K_\Phi} K \to \Omega_{K/k}$, $\{dw_\Lambda\} \cup \{dv_\Phi\}$ are also K_Φ-linearly independent in Ω_{K_Φ}. Since $\{dw_\Lambda\} \cup \{dv_\Phi\}$ generate K_Φ as a field over k, we have

$$\Omega_{K_\Phi/k} = \left(\bigoplus_{\lambda\in\Lambda} K_\Phi dw_\lambda\right) \bigoplus \left(\bigoplus_{\phi\in\Phi} K_\Phi dv_\phi\right). \tag{6.2}$$

The ring $K_\Lambda[V_\Phi]$ is a localization of a polynomial ring over k, hence

$$H_1(k, K_\Lambda[V_\Phi], K_\Phi) = 0. \tag{6.3}$$

By (6.2), (6.3) and Property 2.2 (applied to the surjective map $K_\Lambda[V_\Phi] \to K_\Phi$), the Jacobi–Zariski sequence (2.5) for the triple $k \to K_\Lambda[V_\Phi] \xrightarrow{s} K_\Phi$ takes the form

$$\begin{aligned} 0 \to H_1(k, K_\Phi, K_\Phi) \to \frac{I}{I^2} &\to \Omega_{K_\Lambda[V_\Phi]/k} \otimes K_\Phi \\ &\xrightarrow{ds} \left(\bigoplus_{\lambda\in\Lambda} K_\Phi dw_\lambda\right) \bigoplus \left(\bigoplus_{\phi\in\Phi} K_\Phi dv_\phi\right) \to 0. \end{aligned} \tag{6.4}$$

Since ds is an isomorphism, (6.4) implies that $H_1(k, K_\Phi, K_\Phi) \cong \frac{I}{I^2}$, as desired. $\square$

Corollary 6.8. *Let* $x_1, \dots, x_a$ *be elements of* I. *The elements* $x_1, \dots, x_a$ *form a regular system of parameters of the regular local ring* $K_\Lambda[V_\Phi]_I$ *if and only if the natural images of* $x_1, \dots, x_a$ *in* $H_1(k, K_\Phi, K_\Phi)$ *under the isomorphism of Lemma 6.7 form a basis of* $H_1(k, K_\Phi, K_\Phi)$.

Lemma 6.9. *There is a natural injection* $\iota : H_1(k, K_\Phi, K) \to H_1(k, K, K)$. *If* K *is separable over* K_Φ *(cf. Remark 6.4), then* ι *is an isomorphism.*

Proof. Immediate from the Jacobi–Zariski sequence for the triple $k \to K_\Phi \to K$ (Property 2.18): $0 \to H_1(k, K_\Phi, K) \to H_1(k, K, K) \to H_1(K_\Phi, K, K)$, and Property 2.19. $\square$

Corollary 6.10. *Keep the above notation. Suppose* $\dim_K H_1(k, K, K) < \infty$. *Then* Φ *is a finite set and* $\#\Phi \leq \dim H_1(k, K, K)$. *If, in addition,* K *is separable over* K_Φ, *then* $\#\Phi = \dim H_1(k, K, K)$.

Proof. Immediate from Lemmas 6.7 and 6.9. $\square$

Lemma 6.11. *Let* $k \to K$ *be a field extension. Assume that* $\dim H_1(k, K, K) < \infty$. *Then there exists a subfield* $L \subset K$, *containing* k *and finitely generated over* k, *such that the natural map* $H_1(k, L, K) \to H_1(k, K, K)$ *is an isomorphism. Fix one such* L. *Then for any subfield* $K' \subset K$ *with* $L \subset K'$, *the natural map* $H_1(k, K', K) \to H_1(k, K, K)$ *is an isomorphism. Finally, let* K_0 *be an extension of* k, *contained in* K', *such that the natural map* $\Omega_{K_0/k} \otimes_{K_0} K \to \Omega_{K/k}$ *is injective. Then the natural map* $H_1(K_0, K', K) \to H_1(K_0, K, K)$ *is an isomorphism.*

Proof. Write K as a filtered inductive limit of its subfields which are finitely generated over k: $K = \varinjlim_i K_i$. Since André homology commutes with direct limits ([18, Lemma 3.2] and [2, Chapter III, Proposition 35]), $H_1(k, K, K) = \varinjlim_i H_1(k, K_i, K)$. Hence there exists a subfield $L \subset K$, finitely generated over k, with $H_1(k, L, K) \cong H_1(k, K, K)$. For any field K' such that $L \subset K' \subset K$, the map $H_1(k, L, K) \to H_1(k, K, K)$ factors through $H_1(k, K', K)$, so $H_1(k, K', K) \to H_1(k, K, K)$ is surjective. The injectivity of $H_1(k, K', K) \to H_1(k, K, K)$ is given by the Jacobi–Zariski sequence for $k \to K' \to K$ (Property 2.18), so

$$H_1(k, K', K) \cong H_1(k, K, K), \tag{6.5}$$

as desired. To prove the last statement of the lemma we first note that the map $\Omega_{K_0/k} \otimes_{K_0} K \to \Omega_{K/k}$ factors through $\Omega_{K'/k} \otimes_{K'} K \to \Omega_{K/k}$; this implies that the map $\Omega_{K_0/k} \otimes_{K_0} K \to \Omega_{K'/k} \otimes_{K'} K$ is also injective. Now the last statement of the lemma follows from (6.5) and the commutative diagram

(6.6)
$$\begin{array}{ccccccccc}
H_1(k, K_0, K) & \to & H_1(k, K', K) & \to & H_1(K_0, K', K) & \to & \Omega_{K_0/k} \otimes K & \to & \Omega_{K'/k} \otimes K \\
\| & & \wr\| & & \downarrow & & \| & & \downarrow \\
H_1(k, K_0, K) & \to & H_1(k, K, K) & \to & H_1(K_0, K, K) & \to & \Omega_{K_0/k} \otimes K & \to & \Omega_{K/k}
\end{array}$$

given by the Jacobi–Zariski sequences for the triples $k \to K_0 \to K'$ and $k \to K_0 \to K$, by the five lemma. □

Lemma 6.12. *Let $k \to K_0 \to K$ be a composition of field extensions. Assume that the natural map $\Omega_{K_0/k} \otimes_{K_0} K \to \Omega_{K/k}$ is injective and the natural injection $H_1(k, K_0, K) \to H_1(k, K, K)$ is an isomorphism. Then K is separable over K_0.*

Proof. Immediate from the Jacobi–Zariski sequence for the triple $k \to K_0 \to K$ and Property 2.19. □

Let $\sigma : k \to K$ be a field extension with $\dim H_1(k, K, K) < \infty$. Consider a decomposition of σ of the form (6.1). Although K is unramified over K_Φ, there need not, in general, exist a finitely generated extension of K_Φ, contained in K, over which K is separable. For the proof of Theorem 1.2 we will need to find a decomposition $k \to \tilde{K} \to K$ of σ such that K is separable over $\tilde{K}$ and such that $\tilde{K}$ is the limit of an ascending sequence of finitely generated extensions of K_Φ, contained in K.

Proposition 6.13. *Let $\sigma : k \to K$ be a field extension such that $\dim H_1(k, K, K) < \infty$. Consider a decomposition of σ of the form (6.1). There exists a sequence $K_1 \to K_2 \to \cdots \to K_n \to \cdots$ of finitely generated extensions of K_Φ contained in K, having the following properties.*

1. *For each $i \in \mathbb{N}$, the natural map $H_1(k, K_i, K) \to H_1(k, K, K)$ is an isomorphism.*
2. *Let $\tilde{K} := \lim_{i\to\infty} K_i$. Then K is étale over $\tilde{K}$.*

Proof. If K is separable over K_Φ, put $K_i = \tilde{K} = K_\Phi$ and there is nothing to prove. Suppose K is not separable over K_Φ (in particular, $\operatorname{char} k = p > 0$). Let K_1 be a finitely generated extension of K_Φ, contained in K, such that the natural map $H_1(k, K_1, K) \to H_1(k, K, K)$ is an isomorphism (K_1 exists by Lemma 6.11). We define the K_i recursively as follows. Suppose K_i is defined. Since K is unramified over K_Φ we have $K = K_\Phi K^p$ (Corollary 6.2), so $K_i \subset K_\Phi K^p$. Define K_{i+1} to be a finitely generated extension of K_i, contained in K, such that $K_i \subset K_\Phi K_{i+1}^p$. This defines K_i for all $i \in \mathbb{N}$. Put $\tilde{K} = \lim\limits_{i\to\infty} K_i$. Now (1) holds by definition of K_1 and Lemma 6.11. Since $K_\Phi \subset \tilde{K}$, K is unramified over $\tilde{K}$. By construction, $\tilde{K} = K_\Phi \tilde{K}^p$, so $\tilde{K}$ is unramified over K_Φ (Corollary 6.2). Then the Jacobi–Zariski sequence shows that the natural map $\Omega_{K_\Phi/k} \otimes_{K_\Phi} \tilde{K} \to \Omega_{\tilde{K}/k}$ is surjective; hence so is the map $\Omega_{K_\Phi/k} \otimes_{K_\Phi} K \to \Omega_{\tilde{K}/k} \otimes_{\tilde{K}} K$. Since the inclusion $\Omega_{K_\Phi/k} \otimes_{K_\Phi} K \hookrightarrow \Omega_{K/k}$ factors through $\Omega_{\tilde{K}/k} \otimes_{\tilde{K}} K$, the natural map $\Omega_{\tilde{K}/k} \otimes_{\tilde{K}} K \to \Omega_{K/k}$ is injective. The map $H_1(k, \tilde{K}, K) \to H_1(k, K, K)$ is an isomorphism by Lemma 6.11. Thus K is separable over $\tilde{K}$ by Lemma 6.12. Combining this with the fact that K is unramified over $\tilde{K}$, we get that K is étale over $\tilde{K}$ by Properties 2.4 and 2.19. This completes the proof. □

§7. Residue field extensions induced by formally smooth homomorphisms

Let $\sigma : (A, m, k) \to (B, P, K)$ be a formally smooth local homomorphism of local noetherian rings. Let $\hat{B}$ denote the formal completion of B. In this section we deal with the difficulties coming from the inseparability of the field extension $k \to K$, induced by σ, by proving the following version of the Nica–Popescu theorem (see Corollary 7.9 for the original Nica–Popescu theorem). We construct a local noetherian A-algebra $A^\bullet$, smooth over A, and a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ \downarrow & & \downarrow \\ A^\bullet & \xrightarrow{\sigma^\bullet} & \hat{B} \end{array}$$

such that $\sigma^\bullet$ is formally smooth and induces an isomorphism of the residue field of $A^\bullet$ with K. We may take $\dim\ A^\bullet = \dim A + \dim_K H_1(k, K, K)$ (we will see that any ring $A^\bullet$ having the above properties must be at least of that dimension). Let $m^\bullet$ denote the maximal ideal of $A^\bullet$ and let $q := \dim \frac{\hat{B}}{m^\bullet \hat{B}}$. If we adjoin q independent variables to $A^\bullet$ and send them to a set of q elements of B inducing a regular system of parameters of $\frac{\hat{B}}{m^\bullet \hat{B}}$, the resulting homomorphism will still be formally smooth by Corollary 2.25. In other words, we can always enlarge $A^\bullet$ so that $\dim\ A^\bullet = \dim\ B$. By construction, $A^\bullet$ will be a filtered inductive limit of smooth A-algebras of finite type. The situation is greatly simplified in the special case when the field extension $k \to K$ is finitely generated. We will point out what happens in the finitely generated case in order to give the reader an appreciation of the difficulties which arise when K is not finitely generated over k and the need for the somewhat involved construction of $A^\bullet$, carried out in this section.

Acknowledgement. The results of this section are closely related to those of [12]. Since both our statements and proofs are somewhat different from those of Nica and Popescu, we prefer to give an independent exposition.

Let $B_0 := \frac{B}{mB}$ and let P_0 denote the maximal ideal of B_0. Formal smoothness is preserved by base change [10, Chapter 11, (28.E), p. 201]. Taking base change of σ

by $k = \frac{A}{m}$, we see that the local ring (B_0, P_0, K) is formally smooth (equivalently, geometrically regular—cf. Proposition 2.6) over k. This means that

$$H_1(k, B_0, W) = 0 \text{ for any } B_0\text{-module } W \tag{7.1}$$

(Proposition 2.6). Let $d := \dim B - \dim A = \dim B_0$.

Lemma 7.1. *We have* $\dim H_1(k, K, K) \leq d$.

Proof. Immediate from the Jacobi–Zariski sequence (2.5) for the triple $k \to B_0 \to K$, Property 2.2 and (7.1) (with $W = K$): $0 \to H_1(k, K, K) \to \frac{P_0}{P_0^2}$. □

Consider the residue field extension $k \to K$, induced by σ. Let K_Λ and K_Φ be as in (6.1). Let K_1 be a finitely generated extension of K_Φ, contained in K, such that the natural map

$$H_1(k, K_1, K) \to H_1(k, K, K) \tag{7.2}$$

is an isomorphism (K_1 exists by Lemma 6.11). Pick a basis for $\Omega_{K_1/k}$ of the form $\{dw_{\Lambda_1}\} \cup \{dv_{\Phi_1}\}$ such that $\Lambda_1 \supset \Lambda$, $\Phi_1 \supset \Phi$, the sets $\Lambda_1 \setminus \Lambda$ and Φ_1 are finite and w_{Λ_1} are algebraically independent over k, while v_{Φ_1} are algebraic over $K_{\Lambda_1} = k(w_{\Lambda_1})$. Note that if K is finitely generated over k, we may take $K_1 = K_\Phi = K$, $\Lambda_1 = \Lambda$ and $\Phi_1 = \Phi$ (cf. Remarks 6.4 and 6.6 and Lemma 6.9). We get a decomposition (6.1) for the field extension $k \to K_1$: $k \to K_{\Lambda_1} \to K_{\Phi_1} \to K_1$. Since K_1 is finitely generated over K_{Φ_1} and since $\Omega_{K/K_{\Phi_1}} = 0$, K_1 is separable over K_{Φ_1} (cf. Remarks 6.3 and 6.4). By Lemma 6.9, we get an isomorphism

$$H_1(k, K_{\Phi_1}, K_1) \to H_1(k, K_1, K_1). \tag{7.3}$$

Let $a = \#\Phi_1$. By Corollary 6.10, applied to the field extension $k \to K_1$, and using the fact that K is flat over K_1, we obtain $a = \dim_{K_1} H_1(k, K_1, K_1) = \dim_K H_1(k, K_1, K) = \dim_K H_1(k, K, K)$. Make the identification $\Phi_1 = \{1, \ldots, a\}$. For $\lambda \in \Lambda_1$, let $\mathbf{w}_\lambda$ be any representative of w_λ in B; similarly for $\mathbf{v}_i$, $i \in \Phi_1$. Let $V_{\Phi_1} = (V_1, \ldots, V_a)$ be independent variables. Write $K_{\Phi_1} \cong \frac{K_{\Lambda_1}[V_{\Phi_1}]}{I}$ and let $(G_1, \ldots, G_a)$ be a base of I, constructed in §6 (with Φ replaced by Φ_1), so that $G_i \in K_{\Lambda_1}[V_1, \ldots, V_i]$. Let $\mathbf{G}_i$, $1 \leq i \leq a$, be the representative of G_i in $B[V_i]$ obtained by replacing w_λ by $\mathbf{w}_\lambda$ and v_j by $\mathbf{v}_j$, $j < i$. For $1 \leq i \leq a$, let $x_i := \mathbf{G}_i(\mathbf{v}_i)$. Let $A_1 := A[W_{\Lambda_1}, V_{\Phi_1}]_{P \cap A[W_{\Lambda_1}, V_{\Phi_1}]}$.

Theorem 7.2. (1) *The elements $x_1, \ldots, x_a$ can be extended to a regular system of parameters of B_0.*

(2) *Let $W_{\Lambda_1} = \{W_\lambda \mid \lambda \in \Lambda_1\}$ be independent variables and consider the map $A_1 \to B$ which sends W_λ to $\mathbf{w}_\lambda$ and V_i to $\mathbf{v}_i$. This map is injective and flat.*

Proof. (1) Consider the homomorphism between the triples $k \to K_{\Lambda_1}[V_{\Phi_1}] \to K_{\Phi_1}$ and $k \to B_0 \to K$ (the map $K_{\Lambda_1}[V_{\Phi_1}] \to B_0$ is given by sending W_λ to $\mathbf{w}_\lambda$ and V_i to $\mathbf{v}_i$). By functoriality of André homology, we obtain a commutative diagram of the Jacobi–Zariski sequences:

(7.4)

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & H_1(k, K_{\Phi_1}, K) \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & \frac{I}{I^2} \otimes_{K_{\Phi_1}} K \arrow[d, "\delta"] & & \\
0 \arrow[r] & H_1(k, K, K) \arrow[r, "\beta"] & \frac{P_0}{P_0^2} \arrow[r] & \frac{\Omega_{B_0/k}}{P_0 \Omega_{B_0/k}} \arrow[r] & \Omega_{K/k} \to 0
\end{tikzcd}$$

(here the top row is a part of the Jacobi–Zariski sequence for the triple $k \to K_{\Lambda_1}[V_{\Phi_1}] \to K_{\Phi_1}$ and the K_{Φ_1}-module K, and the bottom row is the Jacobi–Zariski sequence for the triple $k \to B_0 \to K$). Now, α is an isomorphism by Lemma 6.7. The map γ is the composition of the isomorphism $H_1(k, K_{\Phi_1}, K) \cong H_1(k, K_1, K)$ (obtained from (7.3) by tensoring by K) with the isomorphism (7.2) (this follows by the functoriality of André homology); thus γ is an isomorphism. Therefore δ is injective. The elements $\delta(G_i)$ are nothing but the images of the x_i in $\frac{P_0}{P_0^2}$. Since $G_1, \dots, G_a$ are K_{Φ_1}-linearly independent in $\frac{I}{I^2}$, they are K-linearly independent in $\frac{I}{I^2} \otimes_{K_{\Phi_1}} K$, hence $x_1, \dots, x_a$ are K-linearly independent in $\frac{P_0}{P_0^2}$. (1) is proved. Now (2) follows from Corollary 6.8 and Corollary 2.22. This completes the proof of Theorem 7.2. □

Remark 7.3. Suppose that K is finitely generated over k. In that case, $K = K_{\Phi_1}$ and we may take $A^\bullet = A_1$; the A-algebra $A^\bullet$ described in the beginning of this section is already constructed (notice that $A^\bullet$ is already in B; there is no need to pass to completions).

We continue with our construction of $A^\bullet$ in the general case.

Definition 7.4. Let A' be a noetherian A_1-subalgebra of $\hat{B}$ such that $\dim\ A' = \dim\ A + a$. Let $x_1, \dots, x_a$ be as in Theorem 7.2. We say that A' is **unramified** over A_1 if A' is flat over A and $\frac{A'}{mA'}$ is a regular local ring of dimension a with regular parameters $x_1, \dots, x_a$.

If A' is unramified over A_1, then the inclusion $A' \to \hat{B}$ is flat by Corollary 2.22. Also, A_1 is unramified over itself by definition of $x_1, \dots, x_a$.

Proposition 7.5. *Let (A_0, P_0, K_0) and (A_2, P_2, K_2) be two local noetherian $A[W_\Lambda, V_\Phi]_{P \cap A[W_\Lambda, V_\Phi]}$-subalgebras of $\hat{B}$ such that A_0 is formally smooth over A. Assume that there exists a non-negative integer $b \le a$ such that $x_1, \dots, x_b$ induce a regular system of parameters of $\frac{A_0}{mA_0}$. Assume that the inclusion $A_0 \to \hat{B}$ is a formally smooth homomorphism of local rings and that A_2 contains A_1 and is unramified over it (in particular,* $\dim\ A_2 = \dim\ A + a$*). Assume that the map $\Omega_{K_0/k} \otimes_{K_0} K \to \Omega_{K/k}$ is injective. Finally, assume that* $\operatorname{char} K = p > 0$, *that K_2 is either inseparable or purely transcendental over K_0 and that A_2 is a localization of a polynomial ring over A_0 (in finitely many variables). Then there exists a sequence $A_2 \to A_3 \to \cdots$ where each A_i is a localization of a polynomial ring over A_0 in finitely many variables, contained in $\hat{B}$, with the following properties:*

1. $\dim\ A_i = \dim\ A + a$ *for all $i \ge 2$.*
2. *A_i is unramified over A_1 for all $i \ge 2$.*
3. *$\tilde{A} := \bigcup_{i=1}^{\infty} A_i$ is étale over A_0 (in particular, smooth).*
4. *The inclusion $\tilde{A} \to \hat{B}$ is a formally smooth homomorphism of local rings and the induced residue field extension is étale.*

Proof. We construct the A_i recursively as follows. Suppose A_i is constructed. Let K_i denote the residue field of A_i. Let $K_0 \to K_{\Lambda_i} \to K_{\Phi_i} = K_i$ be the decomposition (6.1) for the extension $K_0 \to K_i$, where we can choose $K_{\Phi_i} = K_i$ by Theorem 6.5 and Remark 6.6. Here Λ_i and Φ_i are finite sets and $\#\Phi_i = \dim_{K_i} H_1(K_0, K_i, K_i) = \dim_K H_1(K_0, K_i, K) = \dim_K H_1(K_0, K, K)$ is independent of i (by Lemma 6.11); in fact, diagram (6.6) shows that $\#\Phi_i = a - \dim_{K_0} H_1(k, K_0, K_0)$. Let w_{Λ_i} denote the generators of K_{Λ_i} over K_0 and v_{Φ_i} the generators of K_i over K_{Λ_i}. We will assume, inductively, that A_i is the localization of the polynomial ring $A_0[W_{\Lambda_i}, V_{\Phi_i}]$ at the prime ideal P_i which is, by definition, the kernel of the map $A_0[W_{\Lambda_i}, V_{\Phi_i}] \to K_i$ which maps A_0 to K_0, W_λ to w_λ and V_ϕ to v_ϕ. Since $\Omega_{K/K_0} = 0$, we have $\{w_{\Lambda_i}, v_{\Phi_i}\} \subset K^p K_0$ (Corollary 6.2). Let K_{i+1} be a finitely generated extension of K_0, contained in K, such that

$$\{w_{\Lambda_i}, v_{\Phi_i}\} \subset K_{i+1}^p K_0. \tag{7.5}$$

Consider the decomposition (6.1) $K_0 \to K_{\Lambda_{i+1}} \to K_{\Phi_{i+1}} = K_{i+1}$ of the extension $K_0 \to K_{i+1}$ (we may take $K_{\Phi_{i+1}} = K_{i+1}$ by Theorem 6.5 and Remark 6.6). As usual, let $w_{\Lambda_{i+1}}$ be a set of algebraically independent generators of $K_{\Lambda_{i+1}}$ and $v_{\Phi_{i+1}}$ a set of p-independent generators of $K_{\Phi_{i+1}}$ over $K_{\Lambda_{i+1}}$. Put $A_{i+1} := A_0[W_{\Lambda_{i+1}}, V_{\Phi_{i+1}}]_{P_{i+1}}$. By (7.5), for $\lambda \in \Lambda_i$, there exists $n_\lambda \in \mathbb{N}$, elements $a_{\lambda j} \in K_0$ and polynomials $e_{\lambda j}, h_{\lambda j} \in K_0[W_{\Lambda_{i+1}}, V_{\Phi_{i+1}}]$, $1 \le j \le n_\lambda$, such that the inclusion $K_i \to K_{i+1}$ is given by

$$w_\lambda = \sum_{j=1}^{n_\lambda} a_{\lambda j} \left(\frac{e_{\lambda j}(w_{\Lambda_{i+1}}, v_{\Phi_{i+1}})}{h_{\lambda j}(w_{\Lambda_{i+1}}, v_{\Phi_{i+1}})} \right)^p, \quad \lambda \in \Lambda_i \tag{7.6}$$

(the existence of expressions (7.6) follows from (7.5) because the field $K_{i+1}^p K_0$ equals the K_0-*subalgebra* of K_{i+1}, generated by K_{i+1}^p). We also have the analogous statement for v_ϕ, $\phi \in \Phi_i$; elements $a_{\phi j} \in K_0$ and $e_{\phi j}, h_{\phi j} \in K_0[W_{\Lambda_{i+1}}, V_{\Phi_{i+1}}]$ for $\phi \in \Phi_i$, are defined in the same way. Now the idea is to use the relations (7.6) to lift the inclusion $K_i \to K_{i+1}$ to an inclusion $A_i \to A_{i+1}$. Pick and fix representatives $E_{\lambda j}, H_{\lambda j} \in A_0[W_{\Lambda_{i+1}}, V_{\Phi_{i+1}}]$ of $e_{\lambda j}$ and $h_{\lambda j}$ and similiarly for $E_{\phi j}$ and $H_{\phi j}$. Let $\mathbf{a}_{\lambda j}$ be a representative of $a_{\lambda j}$ in A_0, similarly for $\mathbf{a}_{\phi j}$. By construction, $H_{\lambda j}, H_{\phi j} \notin P_{i+1}$. Define the homomorphism $\iota_i : A_i \to A_{i+1}$ by

$$\begin{aligned} \iota_i(W_\lambda) &= \sum_{j=1}^{n_\lambda} \mathbf{a}_{\lambda j} \left(\frac{E_{\lambda j}}{H_{\lambda j}} \right)^p, \quad \lambda \in \Lambda_i \\ \iota_i(V_\phi) &= \sum_{j=1}^{n_\phi} \mathbf{a}_{\phi j} \left(\frac{E_{\phi j}}{H_{\phi j}} \right)^p, \quad \phi \in \Phi_i. \end{aligned} \tag{7.7}$$

Applying Theorem 7.2 (2) with Λ_1, Φ_1 and B replaced by Λ_i, Φ_i and $\hat{B}$, respectively, we see that $A_i \subset \hat{B}$ and $\hat{B}$ is flat (hence faithfully flat) over A_i for all i. This also proves that all the maps $A_i \to A_{i+1}$ are injective, and that $\tilde{A}$ is an A-subalgebra of $\hat{B}$.

Lemma 7.6. *The A_0-algebra $\tilde{A} := \bigcup_{i=2}^{\infty} A_i$ is étale over A_0.*

Proof. Consider a commutative diagram

$$\begin{array}{ccc} A_0 & \longrightarrow & \tilde{A} \\ \big\downarrow & & \big\downarrow \\ E & \longrightarrow & \frac{E}{N} \end{array} \tag{7.8}$$

where E is an A_0-algebra and N is an ideal of E such that $N^2 = 0$. We want to show that there is a unique lifting $\tilde{A} \to E$ which makes this diagram commutative. It is sufficient to prove that for each $i \ge 2$ there exists a unique lifting

$\tau_i : A_i \to E$ compatible with (7.8). To construct τ_i, take any homomorphism $\tau'_{i+1} : A_0[W_{\Lambda_{i+1}}, V_{\Phi_{i+1}}] \to E$, compatible with (7.8). Since

$$A_i \subset A_0 A_{i+1}^p \tag{7.9}$$

by (7.7) and since $N^p = N^2 = 0$, τ'_{i+1} determines a *unique* lifting $\tau_i : A_i \to E$. $\square$

Taking $E = \frac{B}{P^{j+1}}$ and $N = \frac{P^j}{P^{j+1}}$ in (7.8) and passing to the limit as $j \to \infty$ shows that there exists a unique extension of the homomorphism $A_0 \to \hat{B}$ to $\tilde{A} \to \hat{B}$. We have $\Omega_{\tilde{A}/A_0} = 0$ by (7.9). The map $A_0 \to \hat{B}$ is formally smooth, hence the inclusion $\tilde{A} \to \hat{B}$ is formally smooth (by Proposition 2.6 and Remark 2.10). Let $\tilde{K}$ denote the residue field of $\tilde{A}$. Since $K_1 \subset \tilde{K}$, we have

$$H_1(k, \tilde{K}, K) \cong H_1(k, K, K) \tag{7.10}$$

by Lemma 6.11. By Lemma 6.12, K is separable over $\tilde{K}$. Since $K_\Phi \subset \tilde{K}$ and $\Omega_{K/K_\Phi} = 0$, we have $\Omega_{K/\tilde{K}} = 0$ by the Jacobi–Zariski sequence. Hence K is étale over $\tilde{K}$ (Properties 2.4 and 2.19), as desired.

Next, we show that $\tilde{A}$ is noetherian. Indeed, all the maps $A_i \to \hat{B}$ are faithfully flat. Thus the noetherianity of $\tilde{A}$ is given by the following lemma.

Lemma 7.7. *Let $\{A_i\}$ be an inductive system of noetherian rings together with a faithfully flat map from each A_i to a fixed noetherian ring B. Then $\varinjlim_i A_i$ is again noetherian.*

Proof. Let I_j, $j \in \mathbb{N}$ be an ascending chain of ideals of $\tilde{A}$. Then the chain I_jB stabilizes, say for $j \geq j_0$. The ideal $I_{j_0}B$ is finitely generated, hence there exists i sufficiently large so that $I_{j_0}B = (I_{j_0} \cap A_i)B$. Then for any $i' > i$ and any $j \geq j_0$ we have $I_j \cap A_{i'} \subset (I_jB) \cap A_{i'} = (I_{j_0}B) \cap A_{i'} = (I_{j_0} \cap A_i)B \cap A_{i'} \subset (I_{j_0} \cap A_{i'})B \cap A_{i'} = I_{j_0} \cap A_{i'}$, where the last equality holds by faithful flatness of the map $A_{i'} \to B$. Since this holds for all $i' > i$, we have $I_j = I_{j_0}$ for $j \geq j_0$, as desired. $\square$

Finally, it remains to check that each A_i is unramified over A_1. Since the A_i are localizations of polynomial rings over A_0, they are flat over A_0, hence also over A. It remains to show that $x_1, \dots, x_a$ induce a regular system of parameters of $\frac{A_i}{mA_i}$. It is sufficient to prove that $x_{b+1}, \dots, x_a$ induce a regular system of parameters of $\frac{A_i}{m_0 A_i}$. For each i, write $\frac{A_i}{m_0 A_i} = \frac{K_0[W_{\Lambda_i}, V_{\Phi_i}]}{I_i}$. We proceed by induction on i. For $i = 2$, our statement is true because A_2 was assumed to be unramified over A_1. Suppose the statement is true for i. Consider the commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & H_1(K_0, K_i, K_{i+1}) & \xrightarrow{\alpha_i} & \frac{I_i}{I_i^2} \otimes_{K_i} K_{i+1} \\
\downarrow & & \beta_i \downarrow & & \downarrow \\
0 & \longrightarrow & H_1(K_0, K_{i+1}, K_{i+1}) & \xrightarrow{\alpha_{i+1}} & \frac{I_{i+1}}{I_{i+1}^2}
\end{array}$$

The maps α_i and α_{i+1} are isomorphisms by Lemma 6.7, and β_i is an isomorphism by Lemma 6.11. By the induction assumption and Corollary 6.8, the natural images of $x_{b+1}, \dots, x_a$ in $H_1(K_0, K_i, K_{i+1}) \cong H_1(K_0, K_i, K_i) \otimes_{K_i} K_{i+1}$ form a basis of $H_1(K_0, K_i, K_{i+1})$. Hence their images in $H_1(K_0, K_{i+1}, K_{i+1})$ form a basis of $H_1(K_0, K_{i+1}, K_{i+1})$. Therefore $x_{b+1}, \dots, x_a$ induce a regular system of parameters of $\frac{A_{i+1}}{m_0 A_{i+1}}$, as desired. Proposition 7.5 is proved. $\square$

We will now build up the A-algebra $A^\bullet$ described at the beginning of this section recursively, using transfinite induction on the set of generators of the residue field. We will start with $A_0 = A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}$ and use Proposition 7.5 repeatedly, until we arrive at the A-algebra $A^\bullet$ whose residue field is K. But first, we must check that $A_0 = A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}$ satisfies the hypotheses of Proposition 7.5. Namely, we have to check that $\hat{B}$ is formally smooth over $A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}$.

Lemma 7.8. *The ring $\hat{B}$ is formally smooth over $A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}$.*

Proof. Since $\hat{B}$ is formally smooth over B and B over A, $\hat{B}$ is formally smooth over A. Consider the Jacobi–Zariski sequence for the triple $A \to A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]} \to \hat{B}$:

$$\begin{aligned} &0 \to H_1(A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}, \hat{B}, K) \\ &\to \left(\bigoplus_{\lambda\in\Lambda} K dW_\lambda\right) \oplus \left(\bigoplus_{\phi\in\Phi} K dW_\phi\right) \to \Omega_{\hat{B}/A} \otimes K. \end{aligned} \tag{7.11}$$

Now, $\{dw_\lambda\}_{\lambda\in\Lambda}$ and $\{w_\phi\}_{\phi\in\Phi}$ are all linearly independent in $\Omega_{\hat{B}/A} \otimes K$ because their natural images in $\Omega_{K/A} \equiv \Omega_{K/k}$ are linearly independent by definition. Hence the last arrow in (7.11) is injective, so $H_1(A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}, \hat{B}, K) = 0$ and the lemma is proved. ☐

Corollary 7.9 (the Nica–Popescu theorem [12]). *There exists an increasing sequence A'_i of subrings of $\hat{B}$, each of which is a localization of a polynomial ring in finitely many variables over A, such that $\lim\limits_{i\to\infty} A'_i$ is a local noetherian ring of the same dimension as B, the inclusion $\lim\limits_{i\to\infty} A'_i \to \hat{B}$ is formally smooth and the induced residue field extension is separable.*

Proof. Let $A_0 = A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}$ in Proposition 7.5. Choose an increasing sequence Δ_i of finite subsets of Λ such that for each $\phi \in \Phi_i$, v_ϕ is algebraic over $k(w_{\Delta_i}, v_\Phi, w_{\Lambda_i})$. Put

$$A''_i := A[W_{\Delta_i}, V_\Phi, W_{\Lambda_i}, V_{\Phi_i}]_{P\hat{B}\cap A[W_{\Delta_i},V_\Phi,W_{\Lambda_i},V_{\Phi_i}]}.$$

Extend $x_1, \dots, x_a$ to a set $x_1, \dots, x_d$ which induces a regular system of parameters of $\frac{B}{mB}$ and let $A'_i := A''_i[x_{a+1}, \dots, x_d]_{P\hat{B}\cap A''_i[x_{a+1},\dots,x_d]}$. Then $\dim A'_i = \dim B$. The other conclusions are given by Proposition 7.5. ☐

Theorem 7.10. *There exists a smooth local noetherian $A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}$-algebra $A^\bullet$, mapping to $\hat{B}$, such that:*

(1) *$A^\bullet$ is étale over $A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}$.*
(2) *The homomorphism $A^\bullet \to \hat{B}$ is formally smooth and the induced map*
$$\frac{A^\bullet}{P\cap A^\bullet} \to \frac{B}{P}$$
of residue fields is an isomorphism.
(3) $\dim\ A^\bullet = \dim A + a$.
(4) *$A^\bullet$ is a filtered inductive limit of smooth local $A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}$-algebras $\bar{A}$ essentially of finite type, over which $\hat{B}$ is flat; the elements $x_1, \dots, x_a$ form a regular system of parameters for each of the $\frac{\bar{A}}{m}$.*

Proof. We construct $A^\bullet$ by transfinite induction, using Proposition 7.5. Let $\mathcal{A}$ denote the set of local noetherian subalgebras $\tilde{A}$ of $\hat{B}$ such that:

(1) $A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]} \subset \tilde{A}$.
(2) $\tilde{A}$ is étale over $A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]}$.
(3) $\tilde{A}$ is a filtered inductive limit of smooth A-algebras $\bar{A}$ essentially of finite type, such that each $\frac{\bar{A}}{m\bar{A}}$ is a regular a-dimensional local ring with regular parameters $x_1, \dots, x_a$.
(4) The map $\tilde{A} \to B$ is formally smooth and the induced residue field extension is étale.

$\mathcal{A}$ is partially ordered by inclusion; $\mathcal{A} \neq \emptyset$ by Proposition 7.5, applied to

$$A_0 = A[W_\Lambda, V_\Phi]_{P\cap A[W_\Lambda,V_\Phi]} \quad \text{and}$$
$$A_2 = A_1.$$

Given a subset $\{A_\delta \mid \delta \in \Delta\} \subset \mathcal{A}$, which is totally ordered by inclusion, $\bigcup\limits_{\delta\in\Delta} A_\delta \in \mathcal{A}$ (again, since each A_δ comes with a faithfully flat map to the noetherian ring $\hat{B}$, noetherianity of $\bigcup\limits_{\delta\in\Delta} A_\delta$ is given by Lemma 7.7). By Zorn's lemma, $\mathcal{A}$ contains a maximal element $(A^\bullet, m^\bullet, K^\bullet)$. It remains to prove that $K^\bullet \cong K$. Suppose not. Take an element $t \in K \setminus K^\bullet$. By assumption, K is separable over $K^\bullet$. Hence t is either transcendental or separable algebraic over $K^\bullet$.

Case 1. t is separable algebraic over $K^\bullet$. Let h denote the minimal polynomial of t over $K^\bullet$ and let H be any lifting of h to a polynomial over $A^\bullet$. Put $\tilde{A} := \frac{A^\bullet[T]}{(H)}$. Since H is separable, the inclusion $A^\bullet \to \hat{B}$ extends in a unique way to a map $\tilde{A} \to \hat{B}$ (by the implicit function theorem). It is immediate to verify that $\tilde{A} \in \mathcal{A}$, which contradicts the maximality of $A^\bullet$.

Case 2. t is transcendental over $K^\bullet$. Let $\mathbf{t}$ be any representative of t in B. Then $\mathbf{t}$ is transcendental over $A^\bullet$ by Corollary 2.22, applied to the triple $A^\bullet \to A^\bullet[\mathbf{t}]_{P\cap A^\bullet[\mathbf{t}]} \to \hat{B}$. Let $A_2 := A^\bullet[\mathbf{t}]_{P\cap A^\bullet[\mathbf{t}]}$.

The homomorphism $A^\bullet \to A_2$ satisfies the hypotheses of Proposition 7.5, so there exists an $A^\bullet$-algebra $\tilde{A} \in \mathcal{A}$, containing A_2. This is a contradiction, hence $K^\bullet = K$. $\square$

Corollary 7.11. *There exists a local noetherian A-algebra (A', m'), contained in $\hat{B}$, smooth over A, such that:*

(1) *A' is a filtered inductive limit of smooth finite type A-algebras.*
(2) *$\hat{B}$ is formally smooth over A'.*
(3) *$m'B = P$ (in particular,* $\dim\ A' = \dim\ B$*).*
(4) $\frac{A'}{m'} \cong K$.

Proof. Extend $x_1, \dots, x_a$ to a set $x_1, \dots, x_d$ which induces a regular system of parameters of B_0. Put $A' := A^\bullet[x_{a+1}, \dots, x_d]_{P\cap A^\bullet[x_{a+1},\dots,x_d]}$. The inclusion $\sigma^\bullet : A^\bullet \to \hat{B}$ extends to an inclusion $\sigma' : A' \to \hat{B}$; $\hat{B}$ is formally smooth over A' by Corollary 2.25. Let m' denote the maximal ideal of A'. Then $m'\hat{B} = P\hat{B}$. $\square$

§8. Smoothing of an isolated singularity over a local ring

Let the notation be as in Theorem 1.5 and let $\hat{B}$ denote the P-adic completion of B_P. Combining the results of §§4–7, we obtain a commutative diagram (1.7), satisfying (1)–(8) of Proposition 1.6, with B replaced by $\hat{B}$ (this is explained in more detail below; it proves Theorems 1.5 and 1.2 in the case when (B, P) is local and P-adically complete). In this section we show how to replace $\hat{B}$ by B_P (by P-adic approximation) and in §9—how to descend from B_P to B (delocalization). We start with two observations pertaining to both this and the next section.

Remark 8.1. Suppose $P = (0)$. Then B_P is a field. Then $A_{P\cap A}$ and $\rho_P(C_{P\cap C})$ are domains. Letting $D = \rho(C)$, we get that $H_{D/A} \neq (0)$, so D satisfies the conclusion of Theorem 1.5 and (1.3). From now on, we will assume that $P \neq (0)$, both in this and the next section.

Let $S = S_C(\frac{I}{I^2})$. As before, we will assume that $H_{S/A} \subset P \cap S$ (otherwise we put $D = S$ and Theorem 1.5 and (1.3) are proved (cf. (5.1)). Then, by (5.1), $H_{S/A}B$ is a minimal prime of P. From now on, to simplify the notation, we will replace C by S and assume that there is a presentation $C = \frac{A[u_1,\dots,u_n]}{I}$ such that $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\text{Spec } C\setminus V(H_{C/A})}$ is the trivial vector bundle (cf. Lemma 5.2), both in this section and the next. Note that because of (5.1) and Property 2.16, replacing C by S does not affect condition (3) of Proposition 1.6.

Let $m = P \cap A$. In both this and the next section, we will assume that the homomorphism $\sigma_P : A_m \to B_P$ is formally smooth in the P-adic topology. This is weaker than being regular by Proposition 2.7.

We now state and prove the main result of this section:

Proposition 8.2. *Assume that B is local with maximal ideal P (in particular, $P = \sqrt{H_{C/A}B}$). Then there exists a diagram (1.7) satisfying (1)–(8) of Proposition 1.6 (in particular, Theorems 1.5 and 1.2 hold in this case).*

Proof. Let $\bar{f}_1, \dots, \bar{f}_r \in \Gamma(\text{Spec } C \setminus V(H_{C/A}))$ be sections which freely generate $\frac{\mathcal{I}}{\mathcal{I}^2}\big|_{\text{Spec } C\setminus V(H_{C/A})}$. Note that this property is preserved after a change of base of the form $\otimes_A \bar{A}$, where $\bar{A}$ is essentially of finite type over A.

Let (A', m') be the ring whose existence is asserted in Corollary 7.11 (applied to the formally smooth homomorphism σ_P). Apply Proposition 5.1 to the flat homomorphism $\sigma' : A' \to \hat{B}$ and the finite type A'-algebra $C \otimes_A A'$. We get a commutative diagram (1.7), with B replaced by $\hat{B}$, satisfying (1)–(8) of Proposition 1.6. Moreover, since A' is a filtered inductive limit of local A-algebras $\bar{A}$, smooth and essentially of finite type, over which both A' and $\hat{B}$ are flat (by Corollary 2.22), by Lemma 3.8 we may choose one such A-subalgebra $(\bar{A}, \bar{m})$ such that our diagram (1.7) descends to a diagram of $\bar{A}$-algebras, $\bar{m}B = P$ and (4.13) holds for all the generalized blowings up C_i of $C \otimes_A \bar{A}$ involved in the construction. We obtain a commutative diagram

(8.1)
$$\begin{tikzcd}
\bar{A} \arrow[r, "\bar{\sigma}"] \arrow[d] & \hat{B} & & & \\
C \otimes_A \bar{A} \arrow[r, "\pi^{(L)}"] & C_L \arrow[u, "\rho_L"] \arrow[r, "\pi^{(N)}"] & C_{N+L} \arrow[ul] & \bar{C} \arrow[l, "\lambda"'] \arrow[ull] & C_0 \arrow[l, "\bar{g}"'] \arrow[ulll, "\rho_0"']
\end{tikzcd}$$

satisfying (1)–(8) of Proposition 1.6. The idea is to approximate (8.1) in the P-adic topology to get a diagram (1.7) with B instead of $\hat{B}$. To do this, we will use the following facts, which were proved in the course of the construction of (8.1).

Properties 8.3. (1) *The map* $\pi^{(L)}$ *is the composition of* L *generalized blowings up along* $\bar{m}$. *Here* $L = L_0 L_1$, *where* L_0 *is a positive integer such that*

$$\bar{m}^{L_0} B \subset \bar{m} H_{C/A} B \tag{8.2}$$

and $L_1 \in \mathbb{N}$ *is such that (5.6) holds.*

(2) *The algebra* C_L *is an almost complete intersection over* $\bar{A}$. *Let* $C_L = \frac{\bar{A}[u^{(L)}]}{I_L}$ *be the given presentation of* C_L *and let* $f_1^{(L)}, \dots, f_r^{(L)} \in I_L$ *be as in Definition 1.7. The elements* $f_1^{(L)}, \dots, f_r^{(L)}$ *can be chosen so that their respective images in* $\Gamma\left(\text{Spec } C_L \setminus V(mC_L), \frac{\mathcal{I}_L}{\mathcal{I}_L^2}\right)$ *are* $y(\pi^{(L)})^* \bar{f}_1, \dots, y(\pi^{(L)})^* \bar{f}_r$, *for some* $y \in C_L \setminus (P \cap C_L)$ *(Lemmas 5.4 and 5.5).*

(3) *The map* $\pi^{(N)}$ *is a composition of* N *generalized blowings up along* $\bar{m}$, *where* N *satisfies (1.13) and*

$$\bar{m}^N B^r \subset J(f^{(L)}). \tag{8.3}$$

(4) *Write*

$$C_0 = \frac{\bar{A}[G_1, \dots, G_l]}{(F)}, \tag{8.4}$$

where $F = (F_1, \dots, F_r)$ *are linear homogeneous equations over* $\bar{A}$. *Write* $F_j = \sum\limits_{i=1}^{l} a_{ij} G_i$ *and let* a_j *denote the column* r*-vector with entries* a_{ij}. *Let* $I(F)$ *be the submodule of* A^r *generated by* $a_1, \dots, a_l$. *Then*

$$\bar{m}^{2N} \subset I(F) \tag{8.5}$$

(Lemma 4.5).

(5) *Let* $K_{\bar{A}}$ *denote the kernel of the* $\bar{A}$*-linear map* $\bar{A}^l \to \bar{A}^r$ *given by the matrix* (a_{ij}), $K_{\hat{B}}$ *the kernel of the* $\hat{B}$*-linear map* $\hat{B}^l \to \hat{B}^r$ *given by the same matrix. Let* $\mathbf{g}_i = \rho_0(G_i) \in \hat{B}$ *and let* $\mathbf{g}$ *denote the* l*-vector with entries* $\mathbf{g}_i$, $1 \le i \le l$. *Condition (6) of Proposition 1.6 is equivalent to saying that*

$$\mathbf{g} \in K_{\bar{A}} \hat{B}^l. \tag{8.6}$$

We now P-adically approximate the diagram (8.1). Namely, we will construct a new sequence of generalized blowings up

$$C \otimes_A \tilde{A} \xrightarrow{\tilde{\pi}_1} \tilde{C}_1 \xrightarrow{\tilde{\pi}_2} \dots \xrightarrow{\tilde{\pi}_{N+L}} \tilde{C}_{N+L} \tag{8.7}$$

and

$$\tilde{C}_0 \xrightarrow{\tilde{g}} \tilde{C} \xrightarrow{\tilde{\lambda}} \tilde{C}_{N+L} \tag{8.8}$$

along $\tilde{m}$, where $\tilde{A}$ is, in the sense defined below, a P-adic approximation to $\bar{A}$, and (8.7)–(8.8) are P-adic approximations to the corresponding maps in (8.1). Let J denote the $\bar{A}$-submodule of $\bar{A}^r$ generated by $a_1, \dots, a_l$. By (8.5) and Lemma 3.7, $\bar{m}^t \bar{A}^r \subset J$ for $t \gg 0$. Take $t \in \mathbb{N}$ such that

(1) $P^t \hat{B}^l \cap K_{\hat{B}} \subset P K_{\hat{B}}$.
(2) $t > 2N$.

Next, choose $t' \in \mathbb{N}$ such that

(3) $t' > L + 2t$.
(4) $\bar{m}^{t'} C_L^{n_L} \cap Im(d_L) \subset \bar{m}^{2t} Im(d_L)$

(in the notation of (5.5)). Here (1) and (4) can be achieved by the Artin–Rees lemma. We will now approximate (8.1) to within $P^{t'}$. Condition (1) will be needed to deduce (6) of Proposition 1.6. Condition (2) will be needed to ensure that the hypotheses of Lemma 4.4 hold for the sequence (8.7), and also to prove (7) of Proposition 1.6. (3) will be needed to ensure the hypotheses of Lemma 5.4 and (4) to approximate the elements $f_1^{(L)}, \dots, f_r^{(L)} \in I_L$ to within $2t$.

Let $z_1, \dots, z_k$ be a set of generators of $\bar{m}$. Since $\bar{A}$ is a *local* smooth A-algebra, we may take $\bar{A}$ to be of the form $\bar{A} = \frac{A[V]_{P\cap A[V]}}{(h)}$, where $V = (V_1, \dots, V_s)$, $h = (h_1, \dots, h_q)$ and

$$\det \left| \frac{\partial h_i}{\partial V_j} \right|_{1\le i,j\le q} \bar{A} = \bar{A}. \tag{8.9}$$

Let $v_i = \bar{\sigma}(V_i)$ (cf. (8.1)). For each i, $1 \le i \le k$, let $\beta_i(V) \in A[V]_{P\cap A[V]}$ be a representative of z_i. Without loss of generality, we may assume that $\beta_i(V) \in A[V]$. Let $\tilde{V} = (\tilde{V}_1, \dots, \tilde{V}_s)$ be an s-tuple of independent variables. For each i, $1 \le i \le s$, define $\tilde{z}_i := \beta_i(\tilde{V})$. Let $y = \{y_1, \dots, y_a\} \subset B$ be a set of generators of P. Let $U = \{U_{ij} \mid 1 \le i \le a,\ 1 \le j \le k\}$, $W = \{W_{ij} \mid 1 \le i, j \le a\}$, and $X = \{X_{i\alpha} \mid 1 \le i \le q, \alpha \in \mathbb{N}_0^a, |\alpha| = t'\}$ be independent variables.

Let $\tilde{A}_1$ denote the A-algebra with generators Y, $\tilde{V}$, U, W, X, and relations

$$\begin{aligned} Q_i &= \sum_{j=1}^{k} U_{ij}\tilde{z}_j - \sum_{p=1}^{a} W_{ip}Y_p, && 1 \le i \le a, \\ H_i &= h_i(\tilde{V}_1, \dots, \tilde{V}_s) + \sum_{|\alpha|=t'} Y^\alpha X_{i\alpha} && 1 \le i \le q, \end{aligned} \tag{8.10}$$

and let $\tilde{A} := (\tilde{A}_1)_{P\cap\tilde{A}_1}$. Define the map $\tilde{\sigma} : \tilde{A} \to B$ as follows. For each i, $1 \le i \le s$, choose an element $\tilde{v}_i \in (v_i + P^{t'}\hat{B}) \cap B$. Put $\tilde{\sigma}(Y_i) = y_i$ and $\tilde{\sigma}(\tilde{V}_i) = \tilde{v}_i$. Since $h_i(\tilde{v}) \in P^{t'}$, $\tilde{\sigma}$ extends to a homomorphism $\tilde{\sigma} : \frac{A[Y,\tilde{V},X]}{(H)} \to B$, which agrees with $\bar{\sigma}$ mod $P^{t'}$. Finally, $\bar{m}\hat{B} = P\hat{B}$ by the choice of $\bar{A}$, hence $(\tilde{z})\hat{B} = P\hat{B}$ by Lemma 5.6 (since $(z)\hat{B} = \bar{m}\hat{B} = P\hat{B}$ and $\bar{\sigma}(z_i) - \tilde{\sigma}(\tilde{z}_i) \in P^{t'}\hat{B}$). Hence

$$(\tilde{z})B = P \tag{8.11}$$

by faithful flatness of $\hat{B}$ over B. Put $\tilde{\sigma}(W_{ii}) = 1$. By (8.10)–(8.11), $\tilde{\sigma}$ can be extended to a map $\tilde{\sigma} : \tilde{A} \to B$ (for example, we may take $\tilde{\sigma}(W_{ij}) = 0$ for $i \ne j$). Let $\tilde{m} := P \cap \tilde{A}$. By construction,

$$(Y)\tilde{A} \subset (\tilde{z})\tilde{A} \tag{8.12}$$

(since one can solve for Y in the system of linear equations $Q_1, \dots, Q_a$). On the other hand, $\frac{\tilde{A}}{(Y,\tilde{z})} \cong \frac{A[V]_{P\cap A[V]}}{(h,\beta_1(V),\dots,\beta_k(V))} \cong \frac{\bar{A}}{\bar{m}}$ which is a field. Thus

$$(\tilde{z})\tilde{A} = (\tilde{z}, Y)\tilde{A} = \tilde{m}. \tag{8.13}$$

We have

$$(Y) \subset \Delta_{(Q,H),(W,X)}; \tag{8.14}$$

in particular,

$$(Y) \subset H_{\tilde{A}/A}. \tag{8.15}$$

On the other hand, after localization at $\tilde{m}$ the equations (8.10) become smooth in view of (8.9), so that $H_{\tilde{A}/A} \not\subset \tilde{m}$. Together with (8.15) this means that

$$H_{\tilde{A}/A} B = B. \tag{8.16}$$

Let $C' = \frac{\tilde{A}[\tilde{u}_1, \dots, \tilde{u}_{\tilde{n}}]}{\tilde{I}}$ be an $\tilde{A}$-algebra. Consider a generalized blowing up $\pi_{z,Y^\alpha} : C' \to C'_1$ of C' along $\tilde{m}$, *with generators* $\tilde{z}_1, \dots, \tilde{z}_k$, $\{Y^\alpha \mid \alpha \in \mathbb{N}_0^a,\ |\alpha| = t'\}$. By definition, such a blowing up is described by the equations

$$R_i = \tilde{u}_i - \tilde{c}_i - \sum_{j=1}^{k} \tilde{z}_j \tilde{u}_{ij}^{(1)} - \sum_{|\alpha|=t'} Y^\alpha X_{i\alpha}^{(1)} = 0. \tag{8.17}$$

The key property of this transformation needed below is the fact that the R_i are linear in $X^{(1)}$ and that

$$(Y) \subset \Delta_{R,X^{(1)}} \tag{8.18}$$

(immediate from definitions).

Consider the A-subalgebra $A[\tilde{V}_1, \dots, \tilde{V}_s] \subset \tilde{A}$ (using (8.10), the map γ defined in (8.19) right below and the fact that $a \neq 0$ by Remark 8.1, it is easy to see that the $\tilde{V}$ satisfy no algebraic relations over A in $\tilde{A}$). Define the map

$$\begin{gathered} \gamma : A[\tilde{V}_1, \dots, \tilde{V}_s] \to \bar{A} \qquad \text{by} \\ \gamma(\tilde{V}_i) = V_i. \end{gathered} \tag{8.19}$$

The homomorphisms $\tilde{\sigma}$ and $\bar{\sigma} \circ \gamma$ agree mod $P^{t'}$. Let $\tilde{A}_0 = \frac{A[\tilde{V}]}{(h(\tilde{V}),(\tilde{z})^{t'})}$, $\bar{A}_0 = \frac{\bar{A}}{\bar{m}^{t'}}$. From the equations (8.10), we see that $\tilde{A}_0$ is a subalgebra of $\frac{\tilde{A}}{\tilde{m}^{t'}}$ and that $\frac{\tilde{A}}{\tilde{m}^{t'}}$ is a free $\tilde{A}_0$-module. We have a commutative diagram

$$\begin{array}{ccc} \bar{A}_0 & \xrightarrow{\bar{\sigma} \otimes_{\bar{A}} \bar{A}_0} & \frac{B}{P^{t'}} \\ \uparrow{\scriptstyle \gamma \otimes_{\bar{A}} \bar{A}_0} & & \uparrow{\scriptstyle \tilde{\sigma} \otimes_{\tilde{A}} \tilde{A}_0} \\ \tilde{A}_0 & \longrightarrow & \frac{\tilde{A}}{\tilde{m}^{t'}} \end{array} \tag{8.20}$$

where $\gamma \otimes_{\bar{A}} \bar{A}_0$ is an isomorphism.

Definition 8.4. Let t be a positive integer. Assume that we are given an $\bar{A}$-algebra C_1 and an $\tilde{A}$-algebra $\tilde{C}_1$, with maps $\bar{\rho}_1 : C_1 \to B$ and $\tilde{\rho}_1 : \tilde{C}_1 \to B$. We say that $\tilde{C}_1$ t**-approximates** C_1 if we have a commutative diagram

$$\begin{array}{ccc} \frac{C_1}{\bar{m}^t} & \xrightarrow{\bar{\rho}_1 \otimes_{\bar{A}} \bar{A}_0} & \frac{B}{P^t} \\ \uparrow{\scriptstyle \gamma_1} & & \uparrow{\scriptstyle \tilde{\rho}_1 \otimes_{\tilde{A}} \tilde{A}_0} \\ \tilde{C}_{01} & \xrightarrow{\iota} & \frac{\tilde{C}_1}{\tilde{m}^t} \end{array} \tag{8.21}$$

compatible with (8.20), where $\tilde{C}_{01}$ is an $\tilde{A}_0$-algebra and γ_1 is an isomorphism. Suppose that $\tilde{C}_1$ t-approximates C_1. We say that a $\tilde{C}_1$-module $\tilde{M}$ t-approximates a C_1-module M if there are $\tilde{A}_0$-modules $\tilde{M}_0$ and $\tilde{M}_1$ and homomorphisms $\frac{M}{\bar{m}^t} \xleftarrow{\gamma_M} \tilde{M}_0 \xrightarrow{\iota_M} \tilde{M}_1 \cong \tilde{M}_0 \otimes_{\tilde{C}_{01}} \frac{\tilde{C}_1}{\tilde{m}^t}$, compatible with (8.21), where γ_M is an isomorphism, ι_M identifies $\tilde{M}_0$ with $\tilde{M}_0 \otimes_{\tilde{C}_{01}} 1$, and $\tilde{M}_1$ is a direct summand of $\frac{\tilde{M}}{\tilde{m}^t}$. Suppose $\tilde{M}$ approximates M and let $f \in M$, $\tilde{f} \in \tilde{M}$. We say that $\tilde{f}$ t**-approximates** f if the natural image $\tilde{f}_0$ of $\tilde{f}$ in $\frac{\tilde{M}}{\tilde{m}^t}$ lies in $\iota_M(\tilde{M}_0)$ and $(\gamma_M \circ \iota_M^{-1})(\tilde{f}_0)$ is the image of f in $\frac{M}{\bar{m}^t}$. Consider submodules $M' \subset M$, $\tilde{M}' \subset \tilde{M}$. We say that $\tilde{M}'$ t-approximates M' (as submodules of $\tilde{M}$ and M, respectively) if there exist sets of generators $(f_1, \dots, f_r)$ of $M' + \bar{m}^t M \mod \bar{m}^t M$ and $(\tilde{f}_1, \dots, \tilde{f}_r)$ of $\tilde{M}' + \tilde{m}^t \tilde{M} \mod \tilde{m}^t \tilde{M}$ such that $\tilde{f}_j$ t-approximates f_j, $1 \le j \le r$. A homomorphism $\tilde{\phi} : \tilde{M} \to \tilde{M}'$ of $\tilde{C}_1$-modules t-approximates a homomorphism $\phi : M \to M'$ of C_1-modules if $\tilde{M}$ t-approximates M, $\tilde{M}'$ t-approximates M', $\tilde{\phi}_1(\tilde{M}_1) \subset \tilde{M}_1'$ and ϕ and $\tilde{\phi}$ are compatible with the maps $\gamma_M, \iota_M, \gamma_{M'}$ and $\iota_{M'}$.

Lemma 8.5. *Let $\tilde{\phi} : \tilde{M} \to \tilde{M}'$ be a surjective homomorphism of $\tilde{C}_1$-modules t-approximating a homomorphism $\phi : M \to M'$ of C_1-modules. If $f \in M, \tilde{f}' \in \tilde{M}'$ are such that $\tilde{f}'$ t-approximates $\phi(f)$, then there exists $\tilde{f} \in \tilde{\phi}^{-1}(\tilde{f}')$ which t-approximates f.*

Proof. Straightforward diagram chasing. □

Lemma 8.6. *Let $\tilde{C}_1$ be an $\tilde{A}$-algebra essentially of finite type, t'-approximating an $\bar{A}$-algebra C_1 essentially of finite type.*

(1) *If $\rho_1(C_1) \subset \bar{\sigma}(\bar{A}) + \bar{m}B$, then $\tilde{\rho}_1(\tilde{C}_1) \subset \tilde{\sigma}(\tilde{A}) + \tilde{m}B$.*
(2) *Let $\pi_z : C_1 \to C_2$ be a generalized blowing up along $\bar{m}$ with generators $z_1, \dots, z_k$, as in (4.2), and $\pi_{\tilde{z},Y^\alpha} : \tilde{C}_1 \to \tilde{C}_2$ a generalized blowing up along $\tilde{m}$ with generators $\tilde{z}_1, \dots, \tilde{z}_k$, Y^α, $|\alpha| = t'$, as in (8.17). Choose π_z and $\pi_{\tilde{z},Y^\alpha}$ in such a way that for each i, $1 \le i \le n_1$, the element $\tilde{c}_i$ of (8.17) t'-approximates c_i of (4.2) (in this case, we will say that $\pi_{\tilde{z},Y^\alpha}$ t'-approximates π_z). Then $\tilde{C}_2$ t'-approximates C_2.*
(3) *Let*

$$\begin{gathered} \pi_z' : \bar{A}[u^{(1)}] \to \bar{A}[u^{(2)}] \qquad \textit{and} \\ \tilde{\pi}' : \tilde{A}[\tilde{u}^{(1)}, X^{(1)}] \to \tilde{A}[\tilde{u}^{(2)}, X^{(2)}] \end{gathered} \tag{8.22}$$

be the maps which induce π_z and $\pi_{\tilde{z},Y^\alpha}$, respectively. If an element $\tilde{f} \in \tilde{A}[\tilde{u}^{(1)}, X^{(1)}]$ t'-approximates $f \in \bar{A}[u^{(1)}]$, then $\tilde{\pi}'(\tilde{f})$ t'-approximates $\pi_z'(f)$ (in other words, the relation of t'-approximation is preserved by generalized blowings up which approximate each other). The same statement is true for an element of $\tilde{C}_1$ t'-approximating an element of C_1, and also an element of a free $\tilde{C}_1$-module $\tilde{M}$, t'-approximating an element of a free C_1-module M, t'-approximated by $\tilde{M}$; ditto for a submodule of $\tilde{M}$ t'-approximating a submodule of M.

Proof. Immediate from definitions. □

Let L and N be as in (8.1) and Properties 8.3. Let (8.7) be a sequence of $N+L$ generalized blowings up of $C \otimes_A \tilde{A}$ along $\tilde{m}$, with generators $\tilde{z}_1, \dots, \tilde{z}_k, \{Y^\alpha\}$. In view of (8.19), the $\tilde{A}$-algebra $C \otimes_A \tilde{A}$ t'-approximates the $\bar{A}$-algebra $C \otimes_A \bar{A}$. By Lemma 8.6 (1) and induction on $N+L$, we have that $\tilde{\rho}_i(\tilde{C}_i) \subset \tilde{\sigma}(\tilde{A}) + \tilde{m}B$ for all $i < N+L$, so that such a sequence (8.7) is well defined. Moreover, by Lemma 8.6 (2) and induction on $N+L$, the sequence (8.7) t'-approximates the sequence $\pi^{(N)} \circ \pi^{(L)}$ of (8.1).

Lemma 8.7. *Let $t \in \mathbb{N}$. Let C_1 be an $\bar{A}$-algebra with $\bar{m} \subset Jac(C_1)$ and $\tilde{C}_1$ an $\tilde{A}$-algebra with $\tilde{m} \subset Jac(\tilde{C}_1)$, t-approximating C_1. Let $\tilde{M}$ be a finitely generated $\tilde{C}_1$-module t-approximating a finitely generated $\bar{A}$-module M and $\tilde{J} \subset \tilde{M}$ a submodule t-approximating a submodule $J \subset M$. Assume that $\tilde{M}_1 = \frac{\tilde{M}}{\tilde{m}^t}$ (in the notation of Definition 8.4). Let s be a positive integer such that*

$$s < t. \tag{8.23}$$

If

$$\bar{m}^s M \subset J, \tag{8.24}$$

then $\tilde{m}^s \tilde{M} \subset \tilde{J}$.

Proof. Let $\tilde{M}_0$ be as in Definition 8.4. By definition of approximation, there exist generators $\tilde{f}_1, \dots, \tilde{f}_r$ of $\frac{\tilde{J}}{\tilde{J} \cap \tilde{m}^t \tilde{M}}$ t-approximating generators $f_1, \dots, f_r$ of $\frac{J}{J \cap \bar{m}^t M}$. By definition of approximation, $\tilde{f}_j \in \tilde{M}_0$. By (8.23)–(8.24), $\frac{\bar{m}^s M}{\bar{m}^t M} \subset \frac{J}{\bar{m}^t M}$. Applying the isomorphism γ_M of Definition 8.4, we get $(\tilde{z})^s \tilde{M}_0 \subset (\tilde{f}_1, \dots, \tilde{f}_r)\tilde{M}_0$, so that $\frac{(\tilde{z})^s \tilde{M}}{(\tilde{z})^t \tilde{M}} \equiv \frac{\tilde{m}^s \tilde{M}}{\tilde{m}^t \tilde{M}} \subset (\tilde{f}_1, \dots, \tilde{f}_r)\frac{\tilde{M}}{\tilde{m}^t \tilde{M}}$ (here we are using that $\tilde{M}_1 = \frac{\tilde{M}}{\tilde{m}^t}$, so that $\frac{\tilde{M}}{\tilde{m}^t} \cong \tilde{M}_0 \otimes_{\tilde{C}_{01}} \frac{\tilde{C}_1}{\tilde{m}^t}$). By (8.23) and Nakayama's Lemma, $\tilde{m}^s \tilde{M} \subset \tilde{J}$, as desired. □

Next, we apply the results of §5 to (8.7). By Lemma 8.6 (3) and induction on L, $\tilde{m}^{L_0} H_{C/A}(\tilde{C}_{L_0})_{P \cap \tilde{C}_{L_0}}$ t'-approximates $\bar{m}^{L_0} H_{C/A}(C_{L_0})_{P \cap C_{L_0}}$ (as submodules of $(\tilde{C}_{L_0})_{P \cap \tilde{C}_{L_0}}$ and $(C_{L_0})_{P \cap C_{L_0}}$, respectively). By the choice of L_0 and Lemma 8.7, $\tilde{m}^{L_0}(\tilde{C}_{L_0})_{P \cap \tilde{C}_{L_0}} \subset \tilde{m} H_{C/A}(\tilde{C}_{L_0})_{P \cap \tilde{C}_{L_0}}$. Thus the hypotheses of Lemma 5.4 are satisfied for $\tilde{\pi}_{L_0} \circ \dots \circ \tilde{\pi}_1$, and so $\tilde{C}_L$ is an almost complete intersection at $P \cap \tilde{C}_L$. Pick and fix a set $w_1, \dots, w_b$ of generators of $H^{L_1}_{C/A}$. Write

$$\tilde{C}_i = \frac{\tilde{A}[\tilde{u}^{(i)}_1, \dots, \tilde{u}^{(i)}_{n_i}, X^{(i)}]}{\tilde{I}_i}, \tag{8.25}$$

where $\tilde{u}^{(i)}_j$ t'-approximates $u^{(i)}_j$, $1 \le j \le n_i$. Let k_i be the number of the $X^{(i)}$ variables. Applying Lemma 8.6 (and induction on L) to each $w_s \tilde{d}_0(\bar{f}_j) \in C^n$, we get that its image in $\tilde{C}^n_L$ t'-approximates its image in C^n_L. By the choice of L_1, this implies that $\tilde{z}^\alpha \tilde{d}_0(\bar{f}_j) \in \tilde{C}^n_L$, where $|\alpha| = L$, t'-approximates $z^\alpha \tilde{d}_0(\bar{f}_j) \in C^n_L$. Hence $\tilde{d}_L((\tilde{\pi}^{(L)})^* \bar{f}_j) \in \tilde{C}^{n_L + k_L}_L$ t'-approximates $d_L((\pi^{(L)})^* \bar{f}_j) \in \bar{C}^{n_L}_L$ (note that mod $\tilde{m}^{t'}$ the last k_L components of $\tilde{d}_L((\tilde{\pi}^{(L)})^* \bar{f}_j)$ in $\tilde{C}^{n_L + k_L}_L$ are 0). Moreover, this property does not change after multiplying, respectively, by an element $y \in \bar{C}_L \setminus (P \cap \bar{C}_L)$ and $\tilde{y} \in \tilde{C}_L \setminus (P \cap \tilde{C}_L)$, t'-approximating y. By condition (4) in the definition of t', $\tilde{d}_L((\tilde{\pi}^{(L)})^* \bar{f}_j)$ *as an element of* $\tilde{d}_L\left(\frac{\tilde{I}_L}{\tilde{I}^2_L}\right)$ $(2t)$-approximates $d_L((\pi^{(L)})^* \bar{f}_j)$ as an element of $d_L\left(\frac{I_L}{I^2_L}\right)$. By Lemma 8.5, applied to the $\tilde{C}_L$-module homomorphism $\frac{\tilde{I}_L}{\tilde{I}^2_L} \to \tilde{d}_L\left(\frac{\tilde{I}_L}{\tilde{I}^2_L}\right)$, $(2t)$-approximating the C_L-module homomorphism $\frac{I_L}{I^2_L} \to d_L\left(\frac{I_L}{I^2_L}\right)$, the inverse image $\tilde{\bar{f}}^{(L)}_j$ of $\tilde{d}_L((\tilde{\pi}^{(L)})^* \bar{f}_j)$ in $\frac{\tilde{I}_L}{\tilde{I}^2_L}$ can be chosen to $(2t)$-approximate $\bar{f}^{(L)}_j \in \frac{I_L}{I^2_L}$. Next, apply Lemma 8.5 to the $\tilde{C}_L$-module homomorphism $\tilde{I}_L \to \frac{\tilde{I}_L}{\tilde{I}^2_L}$, $(2t)$-approximating the C_L-module homomorphism $I_L \to \frac{I_L}{I^2_L}$. By Lemma 8.5, the elements $\tilde{f}^{(L)}_1, \dots, \tilde{f}^{(L)}_r \in \tilde{I}_L$ of Lemma 5.5 can be chosen to $(2t)$-approximate $f_1^{(L)}, \dots, f_r^{(L)} \in I_L$ of (8.1). Hence $\tilde{f}_1^{(N+L)}, \dots, \tilde{f}_r^{(N+L)}$ $(2t)$-approximate $f_1^{(N+L)}, \dots, f_r^{(N+L)}$. By (8.3) and Lemma 8.7 (applied to B viewed both as an $\bar{A}$-algebra and an $\tilde{A}$-algebra), $\tilde{m}^N B^r \subset J(\tilde{f}^{(L)})$. Also, $I(\tilde{f}^{(L)})$ $(2t)$-approximates $I(f^{(L)})$. By (8.5) and Lemma 8.7,

$$\tilde{m}^{2N} \subset I(\tilde{f}^{(N+L)}). \tag{8.26}$$

Thus, the hypotheses of Lemma 4.4 hold for $\tilde{C}_{L+N}$, including (4.16). Applying Lemma 4.4, we get a diagram (1.7), satisfying (1)–(5) and (7)–(8) of Proposition 1.6. It remains to prove (6) of Proposition 1.6. This is given by the following lemma.

Since $I(\tilde{f}^{(N+L)})$ $(2t)$-approximates $I(f^{(N+L)})$, they have the same minimal number of generators in view of (8.5) and Nakayama's Lemma. Let $\tilde{a}_1, \dots, \tilde{a}_l$ be generators of $I(\tilde{f}^{(N+L)})$, $(2t)$-approximating $a_1, \dots, a_l$ of (8.4). View $\tilde{f}^{(N+L)}$ as a column r-vector and write $\tilde{f}^{(N+L)} = \sum_{i=1}^{l} \tilde{a}_i \tilde{g}_i$, $g_i \in \tilde{A}[\tilde{u}^{(N+L)}, X^{(N+L)}]$. Let $K_{\tilde{A}}$ (resp. K_B) denote the kernel of the map $\tilde{A}^l \to \tilde{A}^r$ (resp. $B^l \to B^r$) given by the matrix $(\tilde{a}_1, \dots, \tilde{a}_l)$. Let $K_{\bar{A}}$ (resp. $K_{\hat{B}}$) be the kernel of the map $\bar{A}^l \to \bar{A}^r$ (resp. $\hat{B}^l \to \hat{B}^r$) given by $(a_1, \dots, a_l)$. Let $\tilde{\mathbf{g}}_\mathbf{i}$ denote the image of $\tilde{g}_i$ in B and let $\tilde{\mathbf{g}}$ be the column l-vector with entries $\mathbf{g}_j$. By construction, $\tilde{\mathbf{g}} \in K_B$.

Lemma 8.8. *We have $K_B = K_{\tilde{A}} B^l$. In particular, $\tilde{\mathbf{g}} \in K_{\tilde{A}} B^l$.*

Proof. Of course, $K_{\tilde{A}} B^l \subset K_B$. It remains to prove the opposite inclusion. Since $\bar{\sigma}(a_i) - \tilde{\sigma}(\tilde{a}_i) \in P^{2t} \hat{B}^r$ and since $P^t \hat{B}^r \subset I(\tilde{f}^{(N+L)}) B^r$ by (8.11), (8.13), (8.26) and the choice of t, there exists an $l \times l$ invertible matrix U with entries in $\hat{B}$ such that

$$(\sigma'(a_1), \dots, \sigma'(a_l)) U = (\tilde{\sigma}(\tilde{a}_1), \dots, \tilde{\sigma}(\tilde{a}_l)). \tag{8.27}$$

Moreover, we may take U congruent to the identity matrix mod P^t. (8.27) implies that $K_{\bar{A}} \hat{B}^l + P^t \hat{B}^l = K_{\tilde{A}} \hat{B}^l + P^t \hat{B}^l$ and that $K_B \hat{B}^l \subset K_{\hat{B}} + P^t \hat{B}^l$. Since $\hat{B}$ is flat over $\bar{A}$, $K_{\bar{A}} \hat{B}^l = K_{\hat{B}}$. We obtain

$$K_B \hat{B}^l \subset K_{\bar{A}} \hat{B}^l + P^t \hat{B}^l = K_{\tilde{A}} \hat{B}^l + P^t \hat{B}^l. \tag{8.28}$$

Since $K_{\tilde{A}} \hat{B}^l \subset K_B \hat{B}^l$, (8.28) implies that

$$K_B \hat{B}^l \subset K_{\tilde{A}} \hat{B}^l + (P^t \hat{B}^l \cap K_B \hat{B}^l). \tag{8.29}$$

Since $K_B \hat{B}^l$ and $K_{\hat{B}}$ are isomorphic as $\hat{B}$-modules (by (8.27)) and by the choice of t, we have $(P^t \hat{B}^l) \cap (K_B \hat{B}^l) \subset P K_B \hat{B}^l$. Together with (8.29) this implies $K_B \hat{B}^l \subset K_{\tilde{A}} \hat{B}^l + P K_B \hat{B}^l$, so $K_B \hat{B}^l = K_{\tilde{A}} \hat{B}^l$ by Nakayama's lemma. Since the map $B \to \hat{B}$ is faithfully flat, this implies $K_B = K_{\tilde{A}} B^l$, as desired. This completes the proof of Lemma 8.8 and Proposition 8.2. □

Remark 8.9. Let $\tilde{A}_1$ be as in (8.10). By definitions, the sequence (8.7) is actually defined over $\tilde{A}_1$. Multiplying all the $\tilde{f}_j^{(N+L)}$ and the $\tilde{a}_i$ by an element of $\tilde{A}_1 \backslash (P \cap \tilde{A}_1)$ does not affect the proof or the result. Hence the diagram (1.7) of Proposition 8.2 descends to a diagram of $\tilde{A}_1$-algebras, satisfying (1)–(8) of Proposition 1.6.

§9. Smoothing of ring homomorphisms

In this section we prove

Proposition 9.1. *Let the notation be as in Theorem 1.5. Then there exists a commutative diagram (1.7) satisfying (1)–(8) of Proposition 1.6.*

This will complete the proof of Theorem 1.5 and (1.3).

Proof of Proposition 9.1. By Proposition 8.2 and Remark 8.9, we have a diagram (1.7) of $\tilde{A}_1$-algebras, with B replaced by B_P. In order to descend from B_P to B, we need the following delocalization lemma. Let $Y = (Y_1, \dots, Y_a)$, $U = (U_1, \dots, U_s)$ and $X = (X_1, \dots, X_p)$ be sets of independent variables, $F_j \in C[Y, U, X]$, and $F = (F_1, \dots, F_q)$.

Lemma 9.2. *Consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d] & B \arrow[r, "\beta"] & B_P \\ C[Y] \arrow[ur, "\rho_Y"] \arrow[r] & \tilde{C} = \frac{C[Y,U,X]}{(F)} \arrow[ur, "\tilde{\rho}"'] & \end{tikzcd} \tag{9.1}$$

where $(\rho_Y(Y_1), \dots, \rho_Y(Y_a)) = P$. Assume that the F_i are linear in X, that the coefficients of the X_j in F_i do not depend on U and that

$$(Y)\tilde{C} \subset \sqrt{\Delta_{F,X}\tilde{C}}. \tag{9.2}$$

Let E be an independent variable and let $U_j^* = EU_j$. For a positive integer L, put $X_i^*(L) = E^L X_i$. Let $F_i^*(L) = E^L F_i$, where we view $F_i^*(L)$ as a polynomial in U^*, $X^*(L)$ and E over $C[Y]$. Let $C^*(L) := \frac{C[Y,U^*,X^*(L),E]}{(F^*(L))}$; we have the obvious map $\delta : C^*(L) \to \tilde{C}[E]$. Then:

1. *$C^*(L)_E \cong \tilde{C}[E]_E$.*
2. *$(Y)C^*(L) \subset H_{C^*(L)/C}$.*
3. *For L sufficiently large, there exists a map $\rho^* : C^*(L) \to B$, compatible with (9.1), such that $e := \rho^*(E) \notin P$.*

Proof. (1) is obvious. Since the relations F are linear in X, we have $\Delta_{F,X} \equiv \Delta_{F^*,X^*(L)}$ viewed as ideals in $C[Y]$. Since $(Y)\tilde{C} \subset \sqrt{\Delta_{F,X}\tilde{C}}$, we have $(Y)C^*(L) \subset \sqrt{\Delta_{F^*(L),X^*(L)}C^*(L)} \subset H_{C^*(L)/C}$; this proves (2). To prove (3), choose $e \in B \setminus P$ such that $\beta(e)\tilde{\rho}(U), \beta(e)\tilde{\rho}(X) \subset \beta(B)$ and

$$e\, Ker\ \beta = 0. \tag{9.3}$$

For each $L \in \mathbb{N}$, pick $u_i^*, x_j^*(L) \in B$, $1 \le i \le s$, $1 \le j \le p$, such that $\beta(u_i^*) = \beta(e)\tilde{\rho}(U_i)$ and $\beta(x_j^*(L)) = \beta(e)^L \tilde{\rho}(X_j)$. (9.3) implies that

$$e x_j^*(L-1) = x_j^*(L) \tag{9.4}$$

for all $L \ge 2$ and $1 \le j \le p$. Let γ_i denote the degree of F_i viewed as a polynomial in U, $1 \le i \le q$. Take $L \in \mathbb{N}$ such that $L \ge 2$ and $L > \gamma_i$ for $1 \le i \le q$. Define the map $\rho^* : C^*(L) \to B$ by setting $\rho^*(E) = e$, $\rho^*(U_i^*) = u_i^*$, $\rho^*(X_j^*(L)) = x_j^*(L)$. We have $\beta(F_i^*(L-1)(u^*, x^*(L-1))) = \beta(e^{L-1})F_i(\tilde{\rho}(U), \tilde{\rho}(X)) = 0$, so that $F_i^*(L-1)(u^*, x^*(L-1))) \in Ker\ \beta$. Then, using (9.4), $F_i^*(L)(u^*, x^*(L))) =$

$eF_i^*(L-1)(u^*, x^*(L-1))) \in e\ Ker\ \beta = (0)$. This shows that $\rho^* : C^*(L) \to B$ is well defined. Lemma 9.2 is proved. $\square$

Remark 9.3. If A is reduced, the above argument can be modified so that e is not a zero divisor in B. This will be important in §10 where we investigate injectivity of the map $\psi : D \to B$. Indeed, suppose A is reduced. Then so is B. Let Q be the intersection of the minimal primes of B contained in P. Then $Q = Ker\ \beta$. Let $P_1, \dots, P_h$ be the minimal primes of B not contained in P and let $R = \bigcap_{i=1}^{h} P_i$ (if $h = 0$, we adopt the convention $R = B$). Then $R = Ann(Ker\ \beta)$. In the proof of Lemma 9.2, we chose $e \in R \setminus P$. Since $Q \not\subset P_i$ for $1 \le i \le h$, $Q \not\subset \bigcup_{i=1}^{h} P_i$ for $1 \le i \le h$. Hence there exists $b \in Q$ such that $b \notin P_i$ for $1 \le i \le h$. Replacing e by $e + b$, we may assume that e is not a zero divisor and that $e^N\, Ker\, \beta \subset Q^N \subset P^N$ for all $N \in \mathbb{N}$. Since $(Y) \subset \sqrt{\Delta_{F,X}}$, there exists $N \in \mathbb{N}$ such that

$$P^N B^q \equiv (Y)^N B^q \subset \left(\frac{\partial F}{\partial X_1}, \dots, \frac{\partial F}{\partial X_p}\right) \tag{9.5}$$

(where we view the $\frac{\partial F}{\partial X_p}$ as q-vectors). Choosing $L > \gamma_i + N$ in the proof of Lemma 8.2, we get a map $C^* \to \frac{B}{P^N}$ compatible with (9.1). Using (9.5) and the fact that the F_i are linear in the X_j, we lift this map to $\rho^* : C^* \to B$ as desired.

We come back to the proof of Proposition 9.1. First, note that the only requirement on the elements $y_1, \dots, y_a \in B_P$ in the proof of Proposition 8.2 was that they generate PB_P. Thus, without loss of generality, we may assume that $y_1, \dots, y_a \in B$ and that $(y_1, \dots, y_a)B = P$. Let the notation be as in (8.10), (8.17) and (8.25). Write $\tilde{C}_{N+L} = \frac{C[Y, \tilde{V}, U, W, X, X^{(1)}, \dots, X^{(N+L)}, \tilde{u}]}{(Q,H,R)}$. Let $\bar{X} = (X^{(1)}, \dots, X^{(N+L)})$. Let $(W, X, \bar{X})$ play the role of X and $(U, \tilde{V})$ the role of U in Lemma 9.2. (8.14) and (8.18) imply that the hypothesis (9.2) of Lemma 9.2 is satisfied. By Lemma 9.2, we obtain a $C[Y]$-algebra $C^*_{N+L} = \frac{C[Y, V^*, U^*, W^*, X^*, \bar{X}^*, u^*, E]}{(Q^*, H^*, R^*)}$ together with homomorphism $\delta : C^*_{N+L} \to \tilde{C}_{N+L}[E]$, which becomes an isomorphism after localization by E. Let $A^* = \frac{A[Y, V^*, U^*, W^*, X^*, E]}{(Q^*, H^*)}$. The algebra A^* is nothing but the result of the application of Lemma 9.2 to the A-algebra homomorphism $A[Y] \to \frac{A[Y, \tilde{V}, U, W, X]}{(Q,H)}$. In paritcular, we have a map $\alpha : A^* \to \tilde{A}_1$ which induces an isomorphism

$$A^*_E \cong \tilde{A}_1[E]_E; \quad \text{moreover,} \tag{9.6}$$

$$(Y)A^* \subset H_{A^*/A}. \tag{9.7}$$

Next, we delocalize the sequence $\tilde{C}_0 \xrightarrow{\tilde{g}} \tilde{C} \xrightarrow{\tilde{\lambda}} \tilde{C}_{N+L}$. Namely, choose a positive integer S sufficiently large so that there exist $a_i^* \in A^{*r}$ and $g_i^* \in A^*[u^*, \bar{X}^*]$ such that $\alpha(a_i^*) = E^S \tilde{a}_i$, $\delta(g_i^*) = E^S \tilde{g}_i$ and $\sum_{i=1}^{l} a_i^* g_i^* = 0$ in C^*_{N+L}. Let $\mathbf{g}^*$ denote the column l-vector whose ith entry is the image of $\tilde{g}_i$ in B, multiplied by e^S. Since the matrices $(a_1^*, \dots, a_l^*)$ and $(\tilde{a}_1, \dots, \tilde{a}_l)$ differ by a factor of the form E^T, $T \in \mathbb{N}_0$, we have $K_{A^*} B_P^l = K_{\tilde{A}} B_P^l$. By Lemma 8.8, $\tilde{\mathbf{g}} \in K_{\tilde{A}} B_P^l$. Hence by choosing S sufficiently large, we can ensure that

$$\mathbf{g}^* \in K_{A^*} B. \tag{9.8}$$

Put $G_i^* := E^S\tilde{G}_i$, $C_0^* := \frac{A^*[G^*]}{\left(\sum_{i=1}^l a_i^* G_i^*\right)}$ and $C^* := \frac{A^*[u^*,X^*]}{\left(\sum_{i=1}^l a_i^* g_i^*\right)}$. We have the obvious maps : $C_0^* \xrightarrow{g^*} C^* \xrightarrow{\lambda^*} C_{N+L}^*$. We claim that the resulting diagram (1.7) satisfies (1)–(8) of Proposition 1.6. Indeed, (1) follows from (9.7). (2) holds because $Y_i \in m^*$, by definition of Y. (3) is given by (9.7), Lemma 9.2 (2), the definition of Y and the transitivity and base change properties of smoothness. (5) is trivial and (6) is nothing but (9.8). Finally, (4), (7) and (8) are statements about localizations $C^*_{P\cap C^*}$ and $A^*_{m^*}$ and hence follow from Lemma 9.2 (1), (9.6) and the corresponding properties of the diagram given by Proposition 8.2. This completes the proof of Proposition 9.1 and with it Theorems 1.5 and 1.2. □

§10. Smoothing in the category of subalgebras

In this section we give an affirmative answer to Problem 1.3 when A is reduced and $\frac{B}{Q}$ has infinite transcendence degree over $\kappa(A \cap Q)$ for any minimal prime Q of B. We give two counterexamples. The first shows that the hypothesis of infinite transcendence degree is necessary. The second shows that the hypothesis that A is reduced is necessary for the existence of (1.2) such that ψ is injective and $H_{C/A}B \subset \sqrt{H_{D/C}B}$.

Theorem 10.1. *Assume that one of the following holds.*

(1) $\dim\ A = \dim\ B = 0$.
(2) *A is reduced and for each minimal prime Q of B, there exist infinitely many elements $T_1, \dots, T_n, \dots$ in $\frac{B}{Q}$, algebraically independent over $\kappa(A \cap Q)$.*

Consider a diagram (1.1) with ρ injective. Then there exists a diagram (1.2) such that ψ is injective and $H_{C/A}B \subset \sqrt{H_{D/C}B}$.

Remark 10.2. There are two special cases in which the hypotheses of (2) are satisfied. One is when $B = A[[x_1, \dots, x_n]]$, with x_i independent variables, the other when A is essentially of finite type over a field k or $\mathbb{Z}$ and B is the completion with respect to a non-zero ideal $I \subset A$. To see this in the first case, let A_0 be the subring of A generated by 1. Then $A_0[[x_1, \dots, x_n]]$ has cardinality continuum, hence has uncountable transcendence degree over A_0. Since $A_0[[x_1, \dots, x_n]] \otimes_{A_0} A \subset B$, B has uncountable transcendence degree over A. The proof in the second case is similar, except we let A_0 be an essentially finite type algebra over a countable subfield k_0 of k, satisfying $A = A_0 \otimes_{k_0} k$. This raises the question, suggested by B. Teissier, of describing B *explicitly* as a direct limit of its smooth subalgebras of finite type.

Proof of Theorem 10.1. Case (1). Since both A and B can be written as direct products of local artinian rings, it is sufficient to consider the case when B is local. Then B is a complete local ring and Theorem 10.1 follows from Theorem 7.10, since, in this case, the algebra A^* of Theorem 7.10 is isomorphic to B and is a filtered inductive limit of smooth local A-subalgebras of B, essentially of finite type over a polynomial ring over A.

Case (2). Let the notation be as in Theorem 1.5. We wish to prove that if ρ is injective, we may choose D to be a subalgebra of B in Theorem 1.5. The idea is that in §§9–10 D is obtained from C by a composition of two kinds of operations: adjoining algebraically independent elements and taking a finitely generated algebra defined by *linear* (not necessarily homogeneous) equations. Since A is reduced, so are B and C. Let $P_1, \dots, P_k$ denote the minimal primes of B. Let $Q_i := P_i \cap C$. Renumbering the P_i, we may assume that $Q_1, \dots, Q_h$ are the minimal elements among the $Q_1, \dots, Q_k$, $h \leq k$. Then $(0) = \bigcap\limits_{i=1}^{h} Q_i$, so that $Q_1, \dots, Q_h$ are precisely the minimal primes of C. Let K denote the total ring of fractions of C; K is a direct product of fields.

First, suppose P is a minimal prime of B. Since C is reduced, it is non-singular at its generic points. In this case, $P \cap A$ is a minimal prime of A, and $\kappa(P)$ is smooth (i.e. separable) over $\kappa(P \cap A)$. Hence $\kappa(P \cap C)$ is also separable over $\kappa(P \cap A)$. If $P \cap C$ were a minimal prime of C, this would mean that C is smooth over A at $P \cap C$, which contradicts the fact that $H_{C/A} \subset P \cap C$. Thus $P \cap C$ is not a minimal prime of C. In the above notation, we may take $P = P_k$ and $h < k$. Pick an element $t \in \bigcap\limits_{i=1}^{k-1} P_i$ such that t is algebraically independent over C mod P. Let $x_1, \dots, x_l$ be a set of generators of the ideal $P \cap C$. Let T be an independent variable and let $D := \frac{C[T]}{(x_1 T, \dots, x_l T)}$. Define $\psi : D \to B$ by $\psi(T) = t$. The algebraic independence of t mod P implies that ψ is injective. We have $\sqrt{H_{C/A} D} \subset (P \cap C)D = H_{D/C}$. Since $D_{P \cap D} \cong \frac{C_{P \cap C}}{(x) C_{P \cap C}}(T) = \frac{C_{P \cap C}}{(P \cap C) C_{P \cap C}}(T) = \kappa(P \cap C)(T)$, and since $\kappa(P \cap C) \subset \kappa(P)$ are separable over $\kappa(P \cap A)$, we have $H_{D/A} \not\subset P \cap D$. Replace C by D. Iterating this procedure finitely many times, we may assume that $H_{C/A}$ is not contained in any minimal prime of B (hence also in no minimal prime of C).

Lemma 10.3. *Under the assumptions of Theorem 10.1 (2), consider a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[d, "\tau"'] & B \\ C \arrow[r, "\alpha"] \arrow[ur, "\rho"] & C^* \arrow[u, "\rho^*"'] \end{tikzcd}$$

where $C^* = \frac{C[U_1, \dots, U_s]}{(F_1, \dots, F_q)}$ and the F_i are linear in $U_1, \dots, U_s$. Let P be a non-minimal prime ideal of B, such that $P \cap C$ is not a minimal prime of C, and let $t \in \mathbb{N}$. Assume that ρ is injective. Then there exists an ideal J, contained in the C-torsion submodule of C^*, and an injective homomorphism $\rho_1 : \frac{C^*}{J} \to B$ such that, letting π denote the natural map $C^* \to \frac{C^*}{J}$, we have

(1) $\rho = \rho_1 \circ \pi \circ \alpha$.

(2) $\rho^*(U_i) - \rho_1(U_i) \in P^t$ *for* $1 \leq i \leq s$.

Furthermore, $H_{C^*/C} \subset \sqrt{Ann_{C^*} J}$.

Proof. Write $F_i = \sum\limits_{j=1}^{s} b_{ij} U_j - b_{i0}$. Consider the linear map $b : K^s \to K^q$ given by the $q \times s$ matrix b_{ij}, $j \neq 0$. For $1 \leq j \leq h$, let $K_j = \kappa(Q_j)$. Let $r_j = \operatorname{rk}(b_{il})_{\substack{1 \leq i \leq q \\ 1 \leq l \leq s}}$ mod Q_j. Let ρ_j^* denote the map $C^* \otimes_C K_j \to B \otimes_C K_j$ induced by ρ^*. Let b_B denote the map $K^s \otimes_C B \to K^q \otimes_C B$ induced by b. We will show that there exists $v = (v_1, \dots, v_s) \in (Ker\ b_B) \cap P^t B^s$ such that

$$\operatorname{tr.deg}(K_j[\rho_j^*(U_1) + v_1, \dots, \rho_j^*(U_s) + v_s]/K_j) = s - r_j$$

for $1 \leq j \leq h$. We construct $(v_1, \dots, v_s)$ as follows. By definition of r_j, there exists a subset $S_j \subset \{1, \dots, s\}$, of cardinality $s - r_j$, such that, fixing $s - r_j$ arbitrary elements $v_i^{(j)} \in B$, $i \in S_j$, the system of equations $\sum\limits_{m=1}^{s} b_m v_m^{(j)} = 0$ in the unknowns

$v_m^{(j)}$, $m \notin S_j$ has a unique solution in $B \otimes_C K_j$. Choose $v_i^{(j)} \in B$, $i \in S_j$, such that the extension of K_j which they generate in $B \otimes_C K_j$ has transcendence degree $s - r_j$ over $\rho_j^*(C^* \otimes K_j)$. For $i \notin S_j$, let $v_i^{(j)}$ be defined by $\sum\limits_{i=1}^{s} b_i v_i^{(j)} = 0$. Let $v^{(j)}$ denote the s-vector whose ith entry is $v_i^{(j)}$. Multiplying $v^{(j)}$ by a suitable element of C, we may assume that $v_i^{(j)} \in B$ for all i, $1 \leq i \leq s$. For each j, $1 \leq j \leq h$, pick an element $x_j \in (P^t \cap C) \cap (\bigcap\limits_{\substack{1 \leq i \leq h \\ i \neq j}} Q_i \setminus Q_j)$. Define $v = \sum\limits_{j=1}^{h} x_j v^{(j)} \in Ker\ b_B \cap P^t B^s$. Define $\rho_1 : C^* \to B$ by $\rho_1(U_i) = \rho^*(U_i) + v_i$. For each $1 \leq j \leq h$, the map $\rho_{1j} : C^* \otimes_C K_j \to B \otimes_C K_j$, induced by ρ_1, is injective. Let $J = Ker\ \rho_1$. Then $Ann_C J$ is not contained in any minimal prime of C, i.e. J is contained in the C-torsion submodule of C^*. Take a prime ideal $Q \subset C^*$ such that $H_{C^*/C} \not\subset Q$. Then C_Q^* is smooth over C, hence has no C-torsion. Therefore J is annihilated by some element of $C^* \setminus Q$, i.e. $Ann_{C^*} J \not\subset Q$. This proves that $H_{C^*/C} \subset \sqrt{Ann_{C^*} J}$. Lemma 10.3 is proved. $\square$

We go through the proof of Theorem 1.2 step by step and study the injectivity of all the maps to B which appear along the way. Let $C = \frac{A[u_1, \ldots, u_n]}{I}$ be a presentation of C. Consider a diagram (1.1). The first step is to replace C by the symmetric algebra $S = S_C(\frac{I}{I^2})$ (Lemma 5.2). By Lemma 10.3 there exists an ideal J contained in the C-torsion submodule of S and an injective map $\frac{S}{J} \to B$ compatible with ρ. Let $\tilde{S} = \frac{S}{J}$. By Elkik's Lemma (Lemma 5.2), S has a presentation such that the conormal bundle is globally trivial above the smooth locus of S. Since

$$H_{C/A} S \subset H_{S/C} \subset \sqrt{Ann_S J}, \tag{10.1}$$

S and $\tilde{S}$ are isomorphic above the smooth locus of S over C and $H_{C/A}\tilde{S} \subset H_{\tilde{S}/C}$. Then $H_{C/A}\tilde{S} \subset H_{\tilde{S}/A}$ by transitivity of smoothness. If $H_{\tilde{S}/A} B \not\subset P$, we may take $D = \tilde{S}$ and there is nothing more to do. Assume that $H_{\tilde{S}/A} B \subset P$; then P is a minimal prime of $H_{\tilde{S}/A} B$. Hence we may replace S by $\tilde{S}$ in all the subsequent arguments (note: we are not claiming that the conormal bundle of $\tilde{S}$ is free over the smooth locus of $\tilde{S}$; however, by (10.1), it is free away from the locus defined by the ideal $H := H_{C/A}\tilde{S}$, which has the property that $\sqrt{HB_P} = P$—this is what we actually used in the proof of Theorem 1.2). From now on assume that $\frac{I}{I^2}$ is globally free away from the locus defined by an ideal H, such that $\sqrt{HB_P} = P$. By Propositions 9.1 and 1.6 (and by the proof of Proposition 9.1), there exists a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\sigma"] \arrow[d] & B & & & \\
C \arrow[r] & C[Y] \arrow[u, "\rho_y"] \arrow[r] & C[Y, \tilde{V}, E] \arrow[ul, "\rho_v"'] \arrow[r, "\iota"] & C^*_{N+L} \arrow[ull, "\rho^*_{N+L}"'] \arrow[r, "\phi^*"] & D \arrow[ulll]
\end{tikzcd} \tag{10.2}$$

such that $C[Y, \tilde{V}, E]$ is pure transcendental over C (in particular, $C[Y, \tilde{V}, E]$ is defined by zero linear equations over $C[Y]$), C^*_{N+L} is defined by linear equations over $C[Y,\tilde{V},E]$, D by linear equations over C^*_{N+L},

$$(Y) \subset H_{C^*_{N+L}/C[Y,\tilde{V},E]}, \tag{10.3}$$

$$(Y) \subset H_{D/C^*_{N+L}} \quad \text{and} \tag{10.4}$$

$$H_{D/A}B \not\subset P. \tag{10.5}$$

We want to show that D can be chosen so that the map $D \to B$ is injective. First, we show that the map $\rho_y : C[Y] \to B$ can be chosen to be injective. In other words, we have to show that there exists a set $(y_1, \ldots, y_a)$ of generators of P algebraically independent over C. Take a set $(y_1, \ldots, y_a)$ of generators of P. Suppose $\operatorname{tr.deg}(C[y])/C) = b < a$. Say, $y_1, \ldots, y_b$ are algebraically independent over C and $y_{b+1}, \ldots, y_a$ are algebraic over $C[y_1, \ldots, y_b]$. For each i, $1 \le i \le h$, take an element $x_i \in B$, transcendental over $C[y]$ mod P_i. It is easy to construct a $C[y]$-linear combination x of the x_i, which is transcendental over $C[y]$ mod P_i for each i, $1 \le i \le h$. If $b = 0$, pick $w \in (P \setminus (\bigcup\limits_{i=1}^{h} P_i)) \cap \rho(C)$ and define $y_{a+1} := wx$; then $y_{a+1} \in P$ is transcendental over C. Next, assume $b \ge 1$. Let $y'_{b+1} := y_{b+1} + xy_1$. Replacing y_{b+1} by y'_{b+1}, we increase $\operatorname{tr.deg}(C[y]/C)$ by 1. By induction on b, there exists a choice of generators $(y_1, \ldots, y_a)$ which are algebraically independent over C. That is, we may assume that ρ_y is injective. Applying Lemma 10.2 successively to the homomorphisms $C[Y] \to C[Y,\tilde{V},E]$, $\rho_v(C[Y,\tilde{V},E]) \to \frac{C^*_{N+L}}{(Ker\ \rho_v)C^*_{N+L}}$ and $\rho^*_{N+L}(C^*_{N+L}) \to \frac{D}{(Ker\ \rho^*_{N+L})D}$, we get a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C \arrow[r] & D \arrow[u, "\psi"'] \end{tikzcd}$$

Let $J = Ker\ \psi$. Using (10.3)–(10.4), the fact that smoothness is preserved by base change and the last statement of Lemma 10.3, we have

$$(Y)D \subset H_{D/C} \quad \text{and} \tag{10.6}$$

$$(Y)D \subset \sqrt{Ann_D J}. \tag{10.7}$$

Let $D_1 := \frac{D}{J}$. By (10.6) and (10.7),

$$(Y)D_1 \subset H_{D_1/C}. \tag{10.8}$$

Let $R_1, \ldots, R_l$ denote the minimal primes of A, contained in P. Then $R_1A_m, \ldots, R_lA_m$ are precisely the minimal primes of A_m. By regularity of σ, $R_1B_P, \ldots, R_lB_P$ are precisely the minimal primes of B_P. Similarly, since $D_{P\cap D}$ is smooth over A, $R_1D_{P\cap D}, \ldots, R_lD_{P\cap D}$ are the minimal primes of $D_{P\cap D}$. Since $(y)B = P$ is not a minimal prime of B, $\rho_y(Y) \not\subset R_iB_P$ for any i. Then $(Y)D_{P\cap D} \not\subset R_iD_{P\cap D}$ for any i, $1 \le i \le l$. Since $(Y)D_{P\cap D}$ is not contained in any minimal prime of $D_{P\cap D}$, (10.7) implies that $JD_{P\cap D} = 0$. Hence $D_{P\cap D} = (D_1)_{P\cap D_1}$ so $H_{D_1/A}B \not\subset P$ by (10.5) and Lemma 10.3 (2). We have constructed a C-algebra D_1 together with an injective map $\psi_1 : D_1 \to B$, such that $\sqrt{H_{C/A}B} \subset H_{D_1/C}B$ (cf. (10.8)) and $H_{D_1/A}B \not\subset P$ (hence $\sqrt{H_{C/A}B} \subsetneqq H_{D_1/A}B$), as desired. □

The next example shows that Theorem 10.1 may fail if A is not reduced.

Example 10.4. Let k be a field and let $A := \frac{k[x,y,z]_{(x,y,z)}}{(x^2,xy,xz)}$. Let $B := A[[T]]$. Let $C := \frac{A[u]}{(xu)}$ and define the map $\rho : C \to B$ by $\rho(u) = yT$. We want to show that there does not exist a commutative diagram (1.2) such that ψ is injective and $(x,y,z)B \subset \sqrt{H_{D/C}B}$. Suppose such a diagram (1.2) exists. Since $xu = 0$ in D and since $Ann_D x = (Ann_A x)D$ by flatness, we have $\phi(u) \in (x,y,z)D$. Write $\phi(u) = xv_1 + yv_2 + z^n v_3$ in D, where either $v_3 \notin (x,y,z)D$ or $v_3 = 0$. We have $\psi(xv_1) \in (y,z)B \cap (x)B = ((y,z) \cap (x))B = (0)$, so $xv_1 = 0$ in D by the injectivity of ψ. Hence $\phi(u) = yv_2 + z^n v_3$. We claim that $v_3 = 0$. Indeed, we have $yT = y\psi(v_2) + z^n\psi(v_3)$. Hence $z^n\psi(v_3) \in (y)B \cap (z^n)B = (yz^n)B$, so that $\psi(v_3) \in (x,y)B$. Then $\psi(xv_3) = x\psi(v_3) = 0$, so $xv_3 = 0$ and hence $v_3 \in (x,y,z)D$, which implies $v_3 = 0$. We have shown that $\phi(u) \in (y)D$. Therefore for any prime $P \subset D$ with $H_{D/C} \not\subset P$, we have $u \in (y)C_{P\cap C}$, which implies that $Ann_C \frac{(u,y)}{(y)} \not\subset P \cap C$. This means that $H_{D/C} \subset \sqrt{(Ann_C \frac{(u,y)}{(y)})D} = \sqrt{(x,y)D}$, so that $\sqrt{H_{D/C}B} \subset (x,y)B$, which contradicts $(x,y,z)B \subset \sqrt{H_{D/C}B}$.

The next example shows that if B does not have infinite transcendence degree over A, then Theorem 10.1 may fail, even when A is regular. First, consider the analogous question for modules in Lazard's theorem. By an argument similar to that of Lemma 10.3, one shows that for a reduced ring A, a flat module M which has infinite rank over the residue field at every minimal prime of A, is an inductive limit of free finitely generated A-*submodules*. We give an example showing that this is not so without the infinite rank assumption. We then use it to give an example of a regular homomorphism of noetherian rings $\sigma : A \to B$ such that B is not a direct limit of smooth A-subalgebras of finite type.

Example 10.5. Let k_0 be a field and a_{ij}, b_{ij}, c_{ij}, $j \in \mathbb{N}$, $1 \le i \le 3$, independent variables. Let k be the pure transcendental extension of k_0 with the generators a_{ij}, b_{ij}, c_{ij}, $j \in \mathbb{N}$, $1 \le i \le 3$. Let $A := k[x,y,z]_{(x,y,z)}$, with x, y z independent variables. We define the following inductive system:

$$M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{n-1}} M_n \xrightarrow{\phi_n} \cdots \tag{10.9}$$

of free finitely generated A-modules. Put $M_i = A^3$ for all $i \in \mathbb{N}$. Let f_i, g_i, h_i denote the generators of M_i. Let

$$l_{ij} := a_{ij}x + b_{ij}y + c_{ij}z, \tag{10.10}$$

$1 \le i \le 3$, $j \in \mathbb{N}$. Define the homomorphism ϕ_i by $\phi_i(f_i) = l_{1i}f_{i+1} + l_{2i}g_{i+1}$, $\phi_i(g_i) = -l_{3i}f_{i+1} + l_{2i}h_{i+1}$, $\phi_i(h_i) = -l_{3i}g_{i+1} - l_{1i}h_{i+1}$ for $i \in \mathbb{N}$. Let $M := \varinjlim_{i\to\infty} M_i$. M is an inductive limit of free A-modules, hence flat. We want to show that M is not an inductive limit of free finitely generated A-submodules. Suppose the contrary. Since rk $M = 2$, any free submodule of M can have rank at most 2. Hence there exists a constant $n \in \mathbb{N}$ such that, in the notation of (10.9), M is also the limit of the inductive system

$$M_1 \xrightarrow{\psi} A^2 \xrightarrow{\lambda} M_{n+1} \to A^2 \to M_{2n+1} \to \cdots . \tag{10.11}$$

Represent ψ and λ, respectively, by a 2×3 and a 3×2 matrix with entries in A; denote these matrices by Ψ and Λ. Similarly, represent each ϕ_i by a 3×3 matrix Φ_i with entries in A. By construction, $\Lambda\Psi = \Phi_n\Phi_{n-1}\cdots\Phi_1$. Consider the 2×2 submatrix Φ of $(\Phi_n\Phi_{n-1}\cdots\Phi_1)$, obtained by deleting the third row and the third column of $(\Phi_n \Phi_{n-1} \cdots \Phi_1)$. To get a contradiction, it remains to prove that Φ is not a product of two non-invertible 2×2 matrices with entries in A (we may assume that the first matrix is not invertible because $Im(\Lambda\Psi)$ is not contained in $\Lambda\Psi(Af_1 + Ag_1)$, and the second because $Im(\Lambda\Psi)$ is not contained in a submodule of the form $A(f_2 + bh_2) + A(g_2 + ch_2)$). It is sufficient to prove that $\det \Phi$ is an irreducible polynomial in $k[x, y, z]$. The polynomial $\det \Phi$ is homogeneous in x, y and z of degree $2n$. Moreover, the coefficients of $\det \Phi$ are homogeneous polynomials in a_{ij}, b_{ij} and c_{ij} of degree $2n$, quadratic in the variables a_{ij}, b_{ij}, c_{ij} for each fixed j. For each $i \in \mathbb{N}$, let k_i denote the pure transcendental extension of k_0 generated by x, y, z, a_{lj}, b_{lj}, c_{lj}, where $j \in \mathbb{N}$, $j \neq i$ and $1 \leq l \leq 3$.

Lemma 10.6. *The polynomial* $\det \Phi$*, viewed as a polynomial over* k_i*, is an irreducible quadratic form in the nine variables* a_{ji}*,* b_{ji}*,* c_{ji}*,* $1 \leq j \leq 3$*.*

Proof. The polynomial $\det \Phi$ is a quadratic form over k_i in a_{ji}, b_{ji}, c_{ji}, $1 \leq j \leq 3$. Let $\bar{\Phi}_1$ denote the 3×2 matrix given by the first two columns of Φ_1 and $\bar{\Phi}_n$ the 2×3 matrix given by the first two rows of Φ_n.

Lemma 10.7. *Let* $2 < i < n - 1$*. Write* $\bar{\Phi}_n \Phi_{n-1} \cdots \Phi_{i+1} = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \end{pmatrix}$*,* $\Phi_{i-1} \cdots \Phi_2 \bar{\Phi}_1 = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \\ \psi_{31} & \psi_{32} \end{pmatrix}$*. Then* ϕ_{lj}*,* ψ_{lj}*,* a_{li}*,* b_{li}*,* c_{li}*,* x*,* y*,* z *are all algebraically independent over* k_0*.*

Proof. Since each matrix Φ_t involves a different set of variables a_{lt}, b_{lt}, c_{lt}, it is sufficient to prove that each of the sets $\{\phi_{lj}\}$ and $\{\psi_{lj}\}$ *separately* is algebraically independent over $k_0(x, y, z)$. This is proved by straightforward induction on $n - i$ and i, respectively. $\square$

We have

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \end{pmatrix} \begin{pmatrix} l_{1i} & -l_{3i} & 0 \\ l_{2i} & 0 & -l_{3i} \\ 0 & l_{2i} & -l_{1i} \end{pmatrix} \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \\ \psi_{31} & \psi_{32} \end{pmatrix}. \tag{10.12}$$

Lemma 10.6 follows by a direct calculation. $\square$

By Lemma 10.6, if $\det \Phi = fg$ is a factorization of $\det \Phi$, then for each i, $1 \leq i \leq n$, either f or g is independent of a_{li}, b_{li}, c_{li}. In other words, there is a partition $\{1, \ldots, n\} = \Lambda_f \coprod \Lambda_g$, such that f depends only on x, y, z and a_{li}, b_{li}, c_{li} for $i \in \Lambda_f$ (and is a quadratic form in these nine variables for each fixed i) and similarly for g. Say, $1 \in \Lambda_f$. Let $\tilde{\Phi} := \bar{\Phi}_n \Phi_{n-1} \cdots \Phi_2$, and write $\tilde{\Phi} = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \chi \end{pmatrix}$. By definition, $\bar{\Phi}_1 = \begin{pmatrix} l_{11} & -l_{31} \\ l_{21} & 0 \\ 0 & l_{21} \end{pmatrix}$. Hence $\Phi = \begin{pmatrix} \alpha l_{11} + \beta l_{21} & -\alpha l_{31} + \gamma l_{21} \\ \epsilon l_{21} + \delta l_{11} & -\delta l_{31} + \chi l_{21} \end{pmatrix}$. Viewing $\det \Phi$ as a polynomial in a_{i1}, b_{i1} and c_{i1} over $k_1[x, y, z]$, the coefficient of $a_{21} a_{31}$ is $x^2(\alpha\epsilon - \beta\delta)$. By induction on n, $\alpha\epsilon - \beta\delta$ is an irreducible polynomial in $k[x, y, z]$. By (10.12) and Lemma 10.7, $\det \Phi$ is not divisible by x, hence g is not divisible by x. Thus $g = \alpha\epsilon - \beta\delta$ and f depends only on a_{i1}, b_{i1}, c_{i1} and x, y, z. By symmetry we can make the same argument for n instead of 1. This proves that g depends only on a_{in}, b_{in}, c_{in} and x, y, z. Then $n = 2$, in which case $\det \Phi$ is clearly irreducible. This proves that M is not a direct limit of its free finitely generated submodules.

Remark 10.8. In Example 10.5, rk $M = 2$. Since A is a UFD, any flat rank 1 A-module is a direct limit of free finitely generated submodules. However, there exists a normal local ring A which is not a UFD, and an A-module M of rank 1, which is not a direct limit of free A-submodules. Namely, let k_0 be a field. Let x, y, z be independent variables. Consider an infinite number of variables a_j, b_j and c_j, $j \in \mathbb{N}$. For each j, let r_j and s_j denote the roots of the quadratic equation $(a_jX + b_j)^2 - c_j^2X = 0$. Let k denote the extension of k_0 generated by a_j, b_j, c_j, r_j, s_j, $j \in \mathbb{N}$. Let $A := \frac{k[x,y,z]_{(x,y,z)}}{(z^2-xy)}$. We define the following inductive system:

$$M_1 \xrightarrow{\phi_1} M_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} M_n \xrightarrow{\phi_n} \cdots$$

of free finitely generated A-modules. Put $M_i := A^2$ for all $i \in \mathbb{N}$. Let f_i, g_i denote the generators of M_i. Let $l_{1j} = a_jx + b_jy + c_jz$, $l_{2j} = a_j(x - r_jy)$, $l_{3j} = a_j(x - s_jy)$, $l_{4j} := a_jx + b_jy - c_jz$. Define ϕ_i by $\phi_i(f_i) = l_{1i}f_{i+1} + l_{2i}g_{i+1}$, $\phi_i(g_i) = l_{3i}f_{i+1} + l_{4i}g_{i+1}$ for $i \in \mathbb{N}$. Let $M := \lim\limits_{i\to\infty} M_i$. The module M is an inductive limit of free modules, hence flat. Since $l_{1i}l_{4i} = l_{2i}l_{3i}$ by construction, we have rk $M = 1$. To prove that M is not a direct limit of free submodules, represent the homomorphisms ϕ_i by 2×2 matrices Φ_i. If M is a direct limit of free submodules, these submodules must necessarily be of rank 1. Then for some n we can write $\Phi_n \cdots \Phi_1 = \Lambda\Psi$, where Λ and Ψ are a column 2-vector and a row 2-vector, respectively. Let ϕ denote the left uppermost entry of the matrix $\Phi_n \cdots \Phi_1$. It is sufficient to prove that ϕ cannot be written as a product of two elements of A in a non-trivial way. Suppose $\phi = fg$. Since, for each $i \in \{1, \dots, n\}$, ϕ is a linear homogeneous polynomial in a_j, b_j, c_j, r_j, s_j, there exists a partition $\{1, \dots, n\} = \Lambda_f \coprod \Lambda_g$ such that f depends only on x, y, z and a_j, b_j, c_j, r_j, s_j for $j \in \Lambda_f$ and similarly for g. Then one uses induction on n to show that $\#\Lambda_f, \#\Lambda_g \le 1$, which easily leads to a contradiction.

Next we show how Example 10.5 and the example of Remark 10.8 can be used to construct regular homomorphisms $\sigma : A \to B$ such that B is not a direct limit of smooth A-subalgebras of finite type. In the first example we will have $\operatorname{tr.deg}_A B = 2$, in the second $\operatorname{tr.deg}_A B = 1$.

Example 10.9. Let k_0 be a field and let $k = k_0(a, b, c, d)$, where $a = \{a_{ij}\}_{\substack{j\in\mathbb{N} \\ 1\le i\le 3}}$ and similarly for b, c and d. Let $A = k[x, y, z]_{(x,y,z)}$. Let l_{ij} be as in (10.10) and let B denote the ring generated over A by the symbols $\{f_j, g_j, h_j\}_{j\in\mathbb{N}}$ with the relations

$$\begin{aligned} f_i - d_{1i} &= l_{1i}f_{i+1} + l_{2i}g_{i+1}, \\ g_i - d_{2i} &= -l_{3i}f_{i+1} + l_{2i}h_{i+1}, \\ h_i - d_{3i} &= -l_{3i}g_{i+1} - l_{1i}h_{i+1}, \end{aligned} \tag{10.13}$$

localized at the maximal ideal generated by (x, y, z). There is a natural homomorphism $\iota : B \to k[[x, y, z]]$. Indeed, we can use the relations (10.13) recursively in i to construct homomorphisms $B \to \frac{k[x,y,z]}{(x,y,z)^i}$. By (10.13), B is contained in a pure transcendental extension of $k(x, y, z)$ of degree 2, generated by f_1 and g_1. Since f_1 and g_1 are mapped to elements of $k[[x, y, z]]$ algebraically independent over $k(x, y, z)$, ι is injective.

Claim. The ring B is noetherian.

Proof of Claim. For $i \in \mathbb{N}$, let $B_i := \frac{A[f_i,g_i,h_i]_{(x,y,z)}}{(l_{3i}(f_i-d_{1i})+l_{1i}(g_i-d_{2i})+l_{2i}(h_i-d_{3i}))}$, so that $B = \lim_{i\to\infty} B_i$ and $B_i \subset B$ for all $i \in \mathbb{N}$. By (10.13) all the B_i are birational to each other (and hence also to B). Let $m := (x,y,z)B$. By construction, for every $n \in \mathbb{N}$, $\frac{B}{m^n} \cong \frac{k[x,y,z]}{(x,y,z)^n}$, so that $k[[x,y,z]]$ is the m-adic completion of B. It is sufficient to prove that every *prime* ideal of B is finitely generated. Let I be a prime ideal of B and put $I_i := I \cap B_i$. Since m is finitely generated, we may assume that $I \subsetneqq m$. Since B_i is noetherian, I_i is finitely generated. Hence it is sufficient to prove that for i sufficiently large

$$I_i B_{i+1} = I_{i+1}. \tag{10.14}$$

Let $\lambda_{ij} : B_i \to B_j$ denote the birational map between B_i and B_j. There exist elements $v_{0ij}, v_{1ij}, v_{2ij}, v_{3ij} \in B_i$ such that $B_j = B_i\left[\frac{v_{1ij}}{v_{0ij}}, \frac{v_{2ij}}{v_{0ij}}, \frac{v_{3ij}}{v_{0ij}}\right]$. If $v_{0ij} \notin I$ for all $j > i$, (10.14) holds since in that case all the λ_{ij} become isomorphisms after localization by the multiplicative system S generated by the v_{0ij}, $j > i$, so $I_iB_{i+1} = I_i(B_{i+1})_S \cap B_{i+1} = I_{i+1}(B_{i+1})_S \cap B_{i+1} = I_{i+1}$. Now we compute v_{0ij} explicitly from (10.13) and show that we can choose i sufficiently large so that $v_{0ij} \notin I$. By (10.10) and the identity $l_{3,i+1}(f_{i+1}-d_{1,i+1})+l_{1,i+1}(g_{i+1}-d_{2,i+1})+l_{2,i+1}(h_{i+1}-d_{3,i+1}) = 0$, we have $v_{0i,i+1} = l_{1i}l_{1,i+1} - l_{2i}l_{3,i+1} - l_{3i}l_{2,i+1}$, which is a non-degenerate quadratic form in x, y and z. Since $I \subsetneqq m$, there are at most two distinct values of i for which $v_{0i,i+1} \in I$. The Claim is proved. □

Since B is noetherian, it is a regular 3-dimensional local ring with residue field k; hence the inclusion homomorphism $A \to B$ is regular. By (10.13), $\Omega_{B/A}$ is exactly the module $M \otimes_A B$, where M is the module of Example 10.5. If B is a direct limit of smooth A-subalgebras A_λ of finite type, these subalgebras would have transcendence degree two over A. Now assume, in addition, that char $k = 0$. Then the modules of Kähler differentials $\Omega_{(A_\lambda)_{m\cap A_\lambda}/A} \otimes_{A_\lambda} k[[x,y,z]]$ are free $k[[x,y,z]]$-modules, injecting into $\Omega_{B/A} \otimes_B k[[x,y,z]]$. This is impossible by Example 10.5 (replace $k[x,y,z]_{(x,y,z)}$ in Example 10.5 by $k[[x,y,z]]$). Thus B is not a direct limit of smooth A-subalgebras of finite type.

Similarly, one can use the example of Remark 10.8 to construct a regular homomorphism $\frac{k[x,y,z]_{(x,y,z)}}{(z^2-xy)} \to B$, where B is a noetherian A-algebra of transcendence degree 1 over A which is not a direct limit of smooth A-subalgebras of finite type.

§11. Approximation theorems

In this section, we use Theorem 1.1 to deduce the Artin approximation theorem for Henselian pairs, as well as the nested smoothing and nested Artin approximation theorem. First, we use Elkik's Lemma to strengthen Theorem 1.1 (under some additional hypotheses) by requiring that the smooth A-algebra D of (1.2) be étale over a polynomial ring $A[v_1,\dots,v_k]$.

Definition 11.1 (D. Popescu)**.** Let A be a ring and D a smooth A-algebra of finite type over A. Let

$$D = \frac{A[u]}{(f)}, \tag{11.1}$$

where $u = (u_1,\dots,u_n)$ and $f = (f_1,\dots,f_r)$, be a presentation. We say that (11.1) is **standard** if D is a complete intersection (that is, ht $(f) = r$) and after renumbering the f_i and the u_j, $\det\left|\frac{\partial f_i}{\partial u_j}\right|_{1\le i,j\le r}$ is invertible in D. If D admits a standard presentation over A, we will say that D is **standard** over A.

Note: if D is standard, then it is étale over the polynomial ring $A[u_{r+1},\dots,u_n]$.

Proposition 11.2. *Let $\sigma : A \to B$ be a ring homomorphism and D a smooth finite type A-algebra together with an A-algebra homomorphism $\psi : D \to B$. Let J be an ideal of A such that*

$$\psi(D) \subset \sigma(A) + JB \tag{11.2}$$

and every element of $(1+JB)\cap\psi(D)$ is a unit of B. Then there exists a finite rank projective D-module M, an element $x \in D' := S_D M$ and an A-algebra homomorphism $D'_x \to B$, compatible with ψ, such that D'_x is standard over A. Given any presentation $D = \frac{A[u]}{I}$, M can be chosen to be the direct sum of $\frac{I}{I^2}$ with a finite rank free D-module.

Proof. Let $D = \frac{A[u_1,\dots,u_n]}{I}$ be a presentation of D. First, consider the D-algebra $S_D\frac{I}{I^2}$. By Lemma 5.2, $S_D\frac{I}{I^2}$ has a presentation in which the conormal bundle is globally trivial, that is, $S_D\frac{I}{I^2}$ is a complete intersection. Replacing D by $S_D\frac{I}{I^2}$, we may assume that D is a complete intersection, that is, in the new notation, $I = (f_1,\dots,f_r)$, where $r = \operatorname{ht} I$. Under this assumption, we will show that M can be chosen to be a free finite rank D-module.

Let $\frac{\partial f}{\partial u}$ stand for the $r\times n$ matrix $\left(\frac{\partial f_i}{\partial u_j}\right)$. Since D is smooth over A, we have $\Delta_f D = D$. Then there exists an $n\times r$ matrix $G = (g_{ij})_{\substack{1\le i\le n\\1\le j\le r}}$ with entries in D such that

$$\frac{\partial f}{\partial u}G = Id_r \tag{11.3}$$

(where Id_r stands for the $r\times r$ identity matrix). Let $t = (t_1,\dots,t_r)$ be new variables and put $D' = D[t]$. Extend ψ to a map $\psi' : D' \to B$ in an arbitrary way, for example, by sending all the t_j to 0.

Now, for each (i,j), $1\le i\le n$, $1\le j\le r$, pick an element $a_{ij}\in A$ such that

$$\sigma(a_{ij}) \equiv \psi(g_{ij}) \bmod JB \tag{11.4}$$

(this is possible by (11.2)). Let G_0 denote the $n\times r$ matrix $(a_{ij})_{\substack{1\le i\le n\\1\le j\le r}}$. By (11.3) and (11.4), we have

$$\psi\left(\frac{\partial f}{\partial u}\right)\sigma(G_0) \equiv Id_r \bmod JB, \tag{11.5}$$

so $\det(\psi(\frac{\partial f}{\partial u})\sigma(G_0)) \equiv 1 \bmod JB$ and is therefore invertible. Let $x = \det(\frac{\partial f}{\partial u}G_0)$. Since $\psi(x)$ is invertible, ψ' extends to a map $D'_x \to B$. It remains to show that D'_x is standard.

Consider the change of variables

$$\tilde{u}_i = u_i - \sum_{j=1}^{r} a_{ij}t_j, \quad 1\le i\le n. \tag{11.6}$$

Let $\tilde{f}_i(\tilde{u}_1, \ldots, \tilde{u}_n) := f_i(u_1(\tilde{u}), \ldots, u_n(\tilde{u}))$, $1 \le i \le r$. By (11.6) and the chain rule,

$$\det \left| \frac{\partial \tilde{f}_i}{\partial t_j} \right|_{l \le i,j \le r} = \det \left(\frac{\partial f}{\partial u} G_0 \right) = x,$$

which is invertible in D'_x. Thus the Jacobian matrix of $\tilde{f}_1, \ldots, \tilde{f}_r$ has an $r \times r$ invertible minor, hence D'_x is standard over A. This completes the proof. $\square$

Now let (A, I) be a Henselian pair and let $\hat{A}$ denote the I-adic completion of A. Assume that the natural homomorphism $\sigma : A \to \hat{A}$ is regular. This condition holds, for example, whenever A is an excellent ring or, more generally, a G-ring. If A is a local ring, then the requirement that σ be regular is the definition of a G-ring. If A is local and Henselian, σ is regular if and only if A is excellent. We obtain the following general form of the Artin approximation theorem.

Theorem 11.3. *The approximation property holds for the pair (A, I). That is, for a system of algebraic equations with coefficients in A, any solution in $\hat{A}$ can be approximated by a solution in A arbitrarily closely in the I-adic topology.*

Proof. Consider a system of algebraic equations over A. This means that we are given unknowns $u_1, \ldots, u_n$ and a set of equations which we may regard as an ideal $J \subset A[u_1, \ldots, u_n]$. Consider the finitely generated A-algebra $C = \frac{A[u_1, \ldots, u_n]}{J}$. Consider a formal solution, that is, a homomorphism $\hat{\rho} : C \to \hat{A}$. We want to show that there exists a true solution, i.e. a homomorphism $\rho : C \to A$ which agrees with $\hat{\rho}$ mod I^N, where N is an arbitrarily fixed positive integer. By Theorem 1.1 and Proposition 11.2, we may assume that C is an étale extension of the polynomial ring $A[u_{r+1}, \ldots, u_n]$ (where $r = \operatorname{ht} J$). We define the desired solution $\rho : C \to A$ as follows. For each i, $r < i \le n$, pick $x_i \in A$ such that $\hat{\rho}(u_i) - x_i \in I^N \hat{A}$. Put $\rho(u_i) = x_i$. Now, $\frac{C}{(u_{r+1} - x_{r+1}, \ldots, u_n - x_n)}$ is an étale extension of A having a section in $\frac{A}{I^N}$, hence it has a section in A by the Henselian property of (A, I). $\square$

A further generalization of Artin approximation is the nested approximation theorem. The nested version is useful for studying deformations of singularities, as, for example, in the work of Mostowski and Teissier on algebraization of singularities by deformation. We give two different proofs of the nested approximation. In one of them, we follow Bernard Teissier [19] in that we first prove Teissier's "nested smoothing" theorem and then deduce nested approximation as an easy corollary.

Theorem 11.4 (B. Teissier). *Consider a commutative diagram*

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \ldots & \xrightarrow{\alpha_{m-1}} & A_m = A \\
\downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \downarrow & & \downarrow{\scriptstyle \tau_m} \\
C_1 & \xrightarrow{\delta_1} & C_2 & \xrightarrow{\delta_2} & \ldots & \xrightarrow{\delta_{m-1}} & C_m = C \\
\downarrow{\scriptstyle \hat{\rho}_1} & & \downarrow{\scriptstyle \hat{\rho}_2} & & \downarrow & & \downarrow{\scriptstyle \hat{\rho}_m} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \ldots & \xrightarrow{\beta_{m-1}} & B_m = B
\end{array} \tag{11.7}$$

of homomorphisms between noetherian rings, where C_i is of finite type over A_i, $1 \le i \le m$. Let $A'_i := A_i \otimes_{A_{i-1}} B_{i-1}$ ($A'_1 = A_1$ by convention). Assume that A'_i

is noetherian for all i. Assume that the induced homomorphism $\sigma_i : A'_i \to B_i$ is regular for each i. Then (11.7) can be extended to a commutative diagram

(11.8)
$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\tau_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\tau_2"'] & \dots \arrow[r, "\alpha_{m-1}"] \arrow[d] & A_m = A \arrow[d, "\tau_m"] \\
C_1 \arrow[r, "\delta_1"] \arrow[d, "\phi_1"'] & C_2 \arrow[r, "\delta_2"] \arrow[d, "\phi_2"'] & \dots \arrow[r, "\delta_{m-1}"] \arrow[d] & C_m = C \arrow[d, "\phi_m"] \\
D_1 \arrow[r, "\gamma_1"] \arrow[d, "\psi_1"'] & D_2 \arrow[r, "\gamma_2"] \arrow[d, "\psi_2"'] & \dots \arrow[r, "\gamma_{m-1}"] \arrow[d] & D_m = D \arrow[d, "\psi_m"] \\
B_1 2 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & \dots 2 \arrow[r, "\beta_{m-1}"] & B_m = B
\end{tikzcd}$$

such that for each i, $1 \le i \le m$, D_i is smooth of finite type over $D_{i-1} \otimes_{A_{i-1}} A_i$ (for $i = 1$, this condition should read "smooth over A_1"). Let I be an ideal of A and $I_i := I \cap A_i$, $1 \le i \le m$. Assume that for each i every element of $1 + I_i B_i$ is a unit of B_i and that $\frac{A_i}{I_i} \cong \frac{B_i}{I_i B_i}$. Then D_i can be chosen to be standard over $D_{i-1} \otimes_{A_{i-1}} A_i$.

Proof. We proceed by induction on m. For $m = 1$ the result is nothing but Theorem 1.1 together with Proposition 11.2. Assume $m > 1$. First, apply Theorem 1.1 and Proposition 11.2 to the maps $A'_m \xrightarrow{\tau'_m} C \otimes_{A_{m-1}} B_{m-1} \xrightarrow{\hat{\rho}'_m} B$, where $\sigma'_m = \hat{\rho}'_m \circ \tau'_m$ is regular by assumption and $C \otimes_{A_{m-1}} B_{m-1}$ is of finite type over A'_m. We obtain a factorization

$$A'_m \to C \otimes_{A_{m-1}} B_{m-1} \to D' \to B \tag{11.9}$$

of $\hat{\rho}'_m$, where D' is a smooth (resp. standard) A'_m-algebra. Next, let C'_{m-1} be a finite type $\hat{\rho}_{m-1}(C_{m-1})$-subalgebra of B_{m-1} such that the first two homomorphisms in (11.9) are actually defined over C'_{m-1}, so that the first two maps in (11.9) are obtained from a sequence

$$A_m \otimes_{A_{m-1}} C'_{m-1} \to C \otimes_{A_{m-1}} C'_{m-1} \to \bar{D} \to B, \tag{11.10}$$

where $\bar{D}$ is a smooth (resp. standard) $(A \otimes_{A_{m-1}} C'_{m-1})$-algebra of finite type, by tensoring with B_{m-1} over C'_{m-1}. Consider the following commutative diagram:

(11.11)
$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\tau_1"'] & \dots \arrow[r, "\alpha_{m-3}"] \arrow[d] & A_{m-2} \arrow[r, "\alpha_{m-2}"] \arrow[d] & A_{m-1} \arrow[d, "\tau_{m-1}"] \\
C_1 \arrow[r, "\delta_1"] \arrow[d, "\hat{\rho}_1"'] & \dots \arrow[r, "\delta_{m-2}"] \arrow[d] & C_{m-2} \arrow[r] \arrow[d, "\hat{\rho}_{m-2}"] & C'_{m-1} \arrow[d] \\
B_1 \arrow[r, "\beta_1"] & \dots \arrow[r, "\beta_{m-2}"] & B_{m-2} \arrow[r, "\beta_{m-1}"] & B_{m-1}
\end{tikzcd}$$

Apply the induction hypothesis to (11.11). We obtain a commutative diagram

$$
\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\tau_1"'] & \dots \arrow[r, "\alpha_{m-3}"] \arrow[d] & A_{m-2} \arrow[r, "\alpha_{m-2}"] \arrow[d] & A_{m-1} \arrow[d, "\tau_{m-1}"] \\
C_1 \arrow[r, "\delta_1"] \arrow[d, "\phi_1"'] & \dots \arrow[r, "\delta_{m-3}"] \arrow[d] & C_{m-2} \arrow[r] \arrow[d, "\phi_{m-2}"] & C'_{m-1} \arrow[d] \\
D_1 \arrow[r, "\gamma_1"] \arrow[d, "\psi_1"'] & \dots \arrow[r, "\gamma_{m-3}"] \arrow[d] & D_{m-2} \arrow[r, "\gamma_{m-2}"] \arrow[d, "\psi_{m-2}"] & D_{m-1} \arrow[d, "\psi_{m-1}"] \\
B_1 \arrow[r, "\beta_1"] & \dots \arrow[r, "\beta_{m-3}"] & B_{m-2} \arrow[r, "\beta_{m-2}"] & B_{m-1}
\end{tikzcd}
\tag{11.12}
$$

with D_i is smooth of finite type (resp. standard) over $D_{i-1} \otimes_{A_{i-1}} A_i$, $i \leq m-1$. Put $D := \bar{D} \otimes_{C'_{m-1}} D_{m-1}$. Since $\bar{D}$ is smooth (resp. standard) over $A \otimes_{A_{m-1}} C'_{m-1}$, D is smooth (resp. standard) over $A \otimes_{A_{m-1}} D_{m-1}$. We have the map $D \to B$ coming from the tensor product; the map $C \to D$ is the composition $C \to C \otimes_{A_{m-1}} C'_{m-1} \to \bar{D} \to D = \bar{D} \otimes_{C'_{m-1}} D_{m-1}$, where the first and last maps are given by $x \to x \otimes 1$. This completes the construction of (11.8). $\square$

Theorem 11.5. *Consider a diagram (11.7). Let the assumptions be as in Theorem 11.4. Let I be an ideal of A and let $I_i := I \cap A_i$. Let $\hat{A}_i$ denote the I_i-adic completion of A_i, $\hat{B}_i$ the $I_i B_i$-adic completion of B_i. Assume that*

$$\hat{A}_i = \hat{B}_i \quad \text{for all } i,\ 1 \leq i \leq m, \tag{11.13}$$

and that (A_i, I_i) is a Henselian pair for all i. Then there exists a homomorphism $\rho : C \to A$ such that for each j, $1 \leq j \leq m-1$, $\rho(C_j) \subset A_j$. The homomorphism ρ can be chosen to agree with $\hat{\rho}$ up to an arbitrarily specified power of I.

Remarks 11.6. (1) The algebras C_i in the diagram (11.7) can be regarded as nested systems of algebraic equations over A_i with solutions in B_i, which we want to approximate by solutions in A_i.

(2) Let $\{\ \}$ denote convergent power series and $\langle\ \rangle$ Henselization. The special cases of interest in Theorem 11.5 are $A_j = A_1\langle x_1, \dots, x_j\rangle$ and $B_j = \hat{A}_j$ or $B_j = A_1\{x_1, \dots, x_j\}$, where A_1 is equipped with a multiplicative norm in the sense of [11, §45].

(3) If all the A'_i and B_i are G-rings, as in (2) of this remark, the condition that the σ_i be regular homomorphisms is a consequence of (11.13), so it does not need to be specified separately.

First proof of Theorem 11.5. By Theorem 11.4 we may, replacing the C_i by the D_i, assume that each C_i is standard over $A_i \otimes_{A_{i-1}} C_{i-1}$. Fix an integer $N \in \mathbb{N}$. We will denote by ρ_i the restriction to C_i of the hypothetical homomorphism ρ which we want to construct. We will construct the maps $\rho_i : C_i \to A_i$ recursively in i. For $i = 1$, C_1 is standard over A_1 and the map ρ_1 is constructed exactly as in Theorem 11.3. Suppose that ρ_{i-1} has been constructed. Consider the decomposition

$$(C_{i-1} \otimes_{A_{i-1}} A_i) \to (C_{i-1} \otimes_{A_{i-1}} A_i)[v_1, \dots, v_t] \xrightarrow{\epsilon} C_i,$$

where the v_j are algebraically independent over $C_{i-1} \otimes_{A_{i-1}} A_i$ and ϵ is étale. The map $(\alpha_{i-1} \circ \rho_i) : C_{i-1} \to A_i$ induces a map $C_{i-1} \otimes_{A_{i-1}} A_i \to A_i$. We extend it to a map $\rho_i : C_i \to A_i$ in the usual way: first, let $\rho_i(v_j)$ be any element of $(\hat{\rho}_i(v_j) + I_i^N B_i) \cap A_i$, then extend ρ_i to C_i by definition of Henselian. $\square$

We give another proof of Theorem 11.5, under the additional assumption that the homomorphisms α_i are flat.

Second proof of the Nested Approximation Theorem, assuming that the α_i are flat. Again, we use induction on m. Again, consider the homomorphisms $A'_m \to C \otimes_{A_{m-1}} B_{m-1} \to B$. Since (A, I) is a Henselian pair with I-adic completion $\hat{B}$, so is (A'_m, IA'_m). Applying Theorem 11.3, we obtain a homomorphism $\rho' : C \otimes_{A_{m-1}} B_{m-1} \to A'_m$ which agrees with $\hat{\rho}$ mod I^N. We may regard C as a $(C_{m-1} \otimes_{A_{m-1}} A)$-algebra; let $C = \frac{(C_{m-1} \otimes_{A_{m-1}} A)[v_1, \dots, v_l]}{(g_1, \dots, g_s)}$ be a presentation of C over $C_{m-1} \otimes_{A_{m-1}} A$. For each v_i, write $\rho'(v_i) = \sum\limits_{j=1}^{k_i} (a_{ij} \otimes b_{ij})$ with $a_{ij} \in A$ and $b_{ij} \in B_{m-1}$. Let $w = \{w_{ij} \mid 1 \le i \le l,\ 1 \le j \le k_i\}$ be new variables. Let $\sum\limits_j (a_j \otimes b_j)$ stand for the collection $\Big\{ \sum\limits_{j=1}^{k_i} (a_{ij} \otimes b_{ij}) \ \Big|\ 1 \le i \le l \Big\}$ and similarly for $\Big\{ \sum\limits_j (a_j w_j) \Big\} \subset A[w]$. For each q, $1 \le q \le s$, let $h_q(w) := g_q\big(\sum\limits_j (a_j w_j) \big) \in (C_{m-1} \otimes_{A_{m-1}} A)[w]$. Take a finitely generated C_{m-1}-submodule $M \subset C_{m-1} \otimes_{A_{m-1}} A$ which contains all the coefficients of all the polynomials h_q. Since A is assumed flat over A_{m-1}, $C_{m-1} \otimes_{A_{m-1}} A$ is flat over C_{m-1}. Thus the inclusion $M \subset C_{m-1} \otimes_{A_{m-1}} A$ can be factored through a free C_{m-1}-module $F \cong C_{m-1}^r$. Let $\{e_1, \dots, e_r\}$ be a C_{m-1}-basis of F. For $1 \le q \le s$, write $h_q(w) = \sum\limits_{p=1}^{r} \bar{h}_{qp}(w) e_p$, where $\bar{h}_{qp}(w) \in C_{m-1}[w]$. Let $\bar{h} := \{h_{qp} \mid 1 \le q \le s,\ 1 \le p \le r\}$. Put $\bar{C}_{m-1} := \frac{C_{m-1}[w]}{(\bar{h})}$. Define a homomorphism $\bar{\rho} : \bar{C}_{m-1} \to B_{m-1}$ by setting $\bar{\rho}(w_{ij}) := b_{ij}$. We get a commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\tau_1"'] & \dots \arrow[r, "\alpha_{m-3}"] \arrow[d] & A_{m-2} \arrow[r, "\alpha_{m-2}"] \arrow[d] & A_{m-1} \arrow[d] \\
C_1 \arrow[r, "\delta_1"] \arrow[d, "\hat{\rho}_1"'] & \dots \arrow[r, "\delta_{m-3}"] \arrow[d] & C_{m-2} \arrow[r] \arrow[d, "\hat{\rho}_{m-2}"] & \bar{C}_{m-1} \arrow[d, "\bar{\rho}_{m-1}"] \\
B_1 \arrow[r, "\beta_1"] & \dots \arrow[r, "\beta_{m-3}"] & B_{m-2} \arrow[r, "\beta_{m-2}"] & B_{m-1}
\end{tikzcd} \tag{11.14}$$

By the induction assumption, there exists a homomorphism $\rho_{m-1} : \bar{C}_{m-1} \to A_{m-1}$ which agrees with $\bar{\rho}_{m-1}$ modulo I_{m-1}^N and such that $\rho_{m-1}(C_i) \subset A_i$, $1 \le i \le m-1$. Now, letting $\rho(v_i) := \sum_{j=1}^{k_i} a_{ij} \rho_{m-1}(w_{ij})$ defines a map $\rho : C \to A$ which agrees with $\hat{\rho}$ modulo I^N. This completes the proof. $\square$

Appendix. Regular homomorphisms which are not injective

Let $\sigma : A \to B$ be a homomorphism of noetherian rings and let I denote the kernel of σ. The purpose of this Appendix is to show that σ is regular if and only if $Ann(I)B = B$ and the induced homomorphism $\frac{A}{I} \to B$ is regular.

Let $\bar{A} := \frac{A}{I}$ and let $\pi : A \to \bar{A}$ be the natural map.

Theorem A.1. *Let A be a noetherian ring and $I \subset A$ an ideal. Let $\bar{A} := \frac{A}{I}$ and let C be an $\bar{A}$-algebra. The following two conditions are equivalent:*

(1) *C is regular over A.*
(2) *C is regular over $\bar{A}$ and $Ann_A(I)C = C$.*

Note that both (1) and (2) imply that I is contained in a minimal prime of A.

Proof. There is a 1-1 correspondence between the primes of $\bar{A}$ and the primes of A containing I. For any prime $P \subset \bar{A}$,

$$\kappa\left(\frac{\bar{A}}{P}\right) = \kappa\left(\frac{A}{\pi^{-1}(P)}\right),$$

so $\frac{B}{PB}$ is geometrically regular over one of these fields if and only if it is geometrically regular over the other. For a prime P of A not containing I, $\kappa(\frac{A}{P}) \otimes C$ is the zero ring, hence geometrically regular over $\kappa(\frac{A}{P})$ by definition. Hence, to prove Theorem A.1, it is sufficient to prove

Lemma A.2. *Let A be a ring and $I \subset A$ an ideal. Let $\bar{A} := \frac{A}{I}$ and let M be an $\bar{A}$-module. The following two conditions are equivalent:*

(1) *M is A-flat.*
(2) *M is $\bar{A}$-flat and $Ann_A(I)M = M$.*

Note that if A is noetherian, both (1) and (2) imply that I is contained in a minimal prime of A.

Proof. (1) $\Longrightarrow$ (2). Since M is A-flat, it is a direct limit of free A-modules. For each of the free A-modules $A^n \to M$ mapping to M, tensoring both sides with $\bar{A}$ over A, we get that M is a direct limit of free $\bar{A}$-modules. Hence M is $\bar{A}$-flat.

Since M is an $\bar{A}$-module, every element in M annihilates I. Thus to complete the proof of (1) $\Longrightarrow$ (2) it remains to prove

Sublemma A.3. *Let A be a ring, I an ideal of A and M a flat A-module. Then the set of elements in M annihilating I (which we denote by $Ann_M(I)$) is equal to $Ann_A(I)M$.*

Proof. Let a_λ, $\lambda \in \Lambda$, be a set of generators of I. Consider the exact sequence

$$0 \to Ann(I) \to A \to A^\Lambda, \tag{A.1}$$

where A^Λ stands for the *product* of the (possibly infinite) set of copies of A indexed by Λ and the last map in (A.1) sends x to $\{a_\lambda x\}_{\lambda\in\Lambda}$. Tensor (A.1) with M and apply flatness. This completes the proof of (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (1). Now, assume that $M = Ann_A(I)M$ and that M is $\bar{A}$-flat. Then M can be written as a direct limit of free finitely generated $\bar{A}$-modules. We want to deduce that M can also be written as a direct limit of free A-modules. For that it is sufficient to prove that any map $\bar{A} \to M$ can be factored through a free A-module:

$$\bar{A} \to A^n \to M$$

for some $n \in \mathbb{N}$. Giving a map $\rho : \bar{A} \to M$ is the same as giving an element $x \in M$ (where $x = \rho(1)$). Since $M = Ann_A(I)M$, there exist $y_1, \ldots, y_n \in Ann_A(I)$ and

$x_1, \dots, x_n \in M$ such that

$$x = \sum_{i=1}^{n} y_i x_i. \tag{A.2}$$

Then ρ factors through A^n, where the map $\bar{A} \to A^n$ is given by sending 1 to $\sum_{i=1}^{n} y_i X_i$ (the X_i being the generators of A^n) and $A^n \to M$ sends X_i to x_i. □

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